

# Finite $p$ -groups all of whose subgroups of index $p^3$ are abelian \*

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## Abstract

Suppose that  $G$  is a finite  $p$ -group. If all subgroups of index  $p^t$  of  $G$  are abelian and at least one subgroup of index  $p^{t-1}$  of  $G$  is not abelian, then  $G$  is called an  $\mathcal{A}_t$ -group. In this paper, some information about  $\mathcal{A}_t$ -groups are obtained and  $\mathcal{A}_3$ -groups are completely classified. This solves an *old problem* proposed by Berkovich and Janko in their book [8]. Abundant information about  $\mathcal{A}_3$ -groups are given.

**Keywords** finite  $p$ -groups, minimal non-abelian  $p$ -groups,  $\mathcal{A}_t$ -groups.

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## 1 Introduction

Finite  $p$ -groups are an important class of finite groups. After the classification of finite simple groups was finally completed, the study of finite  $p$ -groups becomes more and more active. Many leading group theorists, for example, Glauberman and Janko have turned their attentions to the study of finite  $p$ -groups. Although finite simple groups are classified, it is impossible to classify finite  $p$ -groups in the classical sense. The reason is that a finite  $p$ -group has “too many” normal subgroups and consequently there is an extremely large number of non-isomorphic  $p$ -groups of a given fixed order. In fact, Higman[15, 16] gave a formula for the number  $f(n, p)$  of non-isomorphic  $p$ -groups of order  $p^n$ :

$$\text{when } n \rightarrow \infty, \quad f(n, p) = p^{n^3(2/27 + O(n^{-1/3}))}.$$

It is easy to see that when  $n$  becomes large, the number of non-isomorphic  $p$ -groups of order  $p^n$  becomes large in exponent speed. For example, up to now, the known results about the classification of 2-groups [11] are:

$n$	2	$2^2$	$2^3$	$2^4$	$2^5$	$2^6$	$2^7$	$2^8$	$2^9$	$2^{10}$
$f(n, 2)$	1	2	5	14	51	267	2328	56092	10494213	49487365422

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For  $p > 2$ ,  $p$ -groups of order  $p^7$  are classified by [21]. The result is  $f(7, 3) = 9310$ ,  $f(7, 5) = 34297$ . For  $p > 5$ ,  
 $f(7, p) = 3p^5 + 12p^4 + 44p^3 + 170p^2 + 707p + 2455 + (4p^2 + 44p + 291) \gcd(p-1, 3) + (p^2 + 19p + 135) \gcd(p-1, 4) + (3p + 31) \gcd(p-1, 5) + 4 \gcd(p-1, 7) + 5 \gcd(p-1, 8) + \gcd(p-1, 9)$ .

Because of the difficult of the classification of finite  $p$ -groups in the classical sense, Janko and Berkovich sponsored and led an research project that aims at classifying certain classes of finite  $p$ -groups defined by their subgroup structure. As Janko mentioned in the Foreword of [7], to study  $p$ -groups with “large” abelian subgroups is another approach to finite  $p$ -groups. We know that non-abelian  $p$ -groups with “largest” abelian subgroups are minimal non-abelian groups. A non-abelian group  $G$  is said to be *minimal non-abelian* if every proper subgroup of  $G$  is abelian. Minimal non-abelian groups were classified in [20], and in more detail for finite  $p$ -groups in [25]. Berkovich and Janko in [6] introduced a new concept,  $\mathcal{A}_t$ -groups, which is a more general concept than that of minimal non-abelian  $p$ -groups. For a positive integer  $t$ , a finite  $p$ -group  $G$  is called an  $\mathcal{A}_t$ -group if all subgroups of index  $p^t$  of  $G$  are abelian, and at least one subgroup of index  $p^{t-1}$  of  $G$  is not abelian. In this paper, an  $\mathcal{A}_0$ -group is an abelian group. Obviously,  $\mathcal{A}_1$ -groups are exactly the minimal non-abelian  $p$ -groups. For small  $t$ ,  $\mathcal{A}_t$ -groups can be considered as groups having “large” abelian subgroups. Many scholars studied and classified  $\mathcal{A}_2$ -groups, see [6, 8, 14, 18, 26, 31]. Obviously, classifying  $\mathcal{A}_3$ -groups is a fascinating problem. In fact, this is a problem proposed by Berkovich and Janko in their joint book [8].

**Problem 1278.** (*Old problem*) *Classify  $\mathcal{A}_3$ -groups.*

In this paper, we completely classify  $\mathcal{A}_3$ -groups in classical sense. The groups described in the title are the totality of abelian groups,  $\mathcal{A}_1$ -groups,  $\mathcal{A}_2$ -groups and  $\mathcal{A}_3$ -groups. Thus the groups described in the title are completely classified.

Related to  $\mathcal{A}_3$ -groups, Berkovich and Janko proposed the following

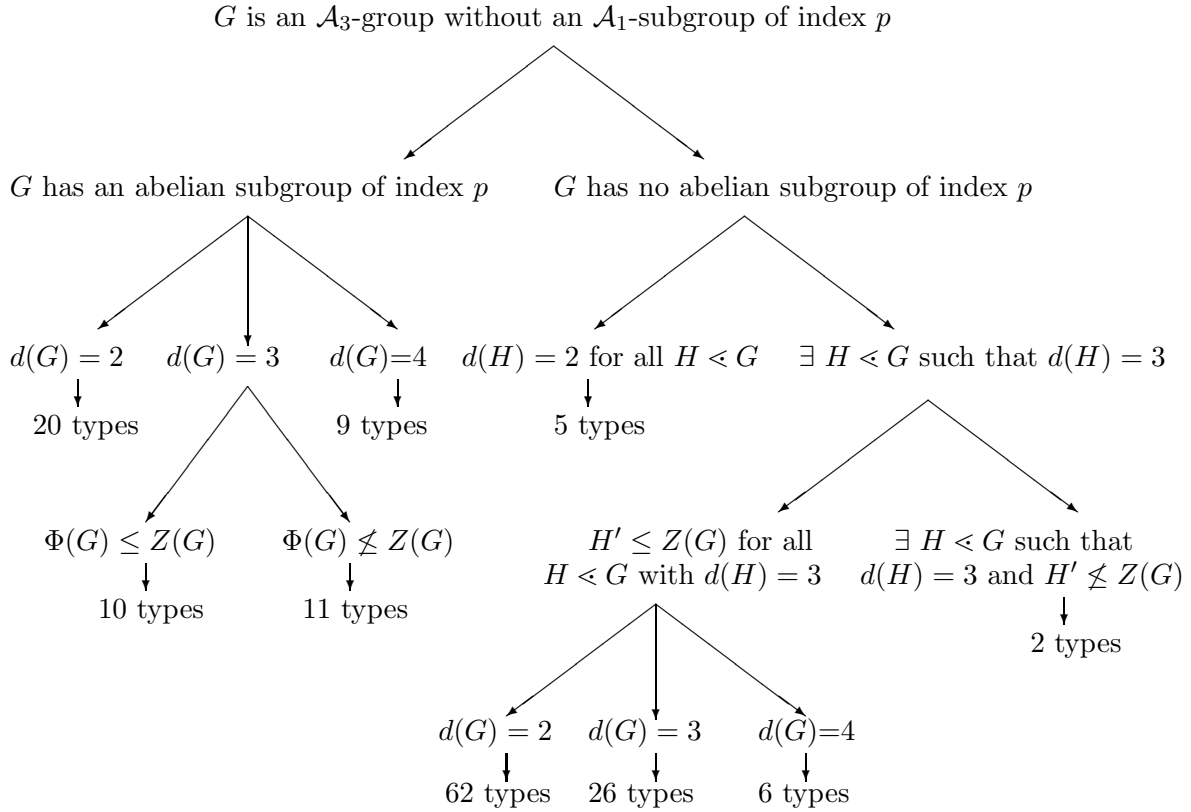
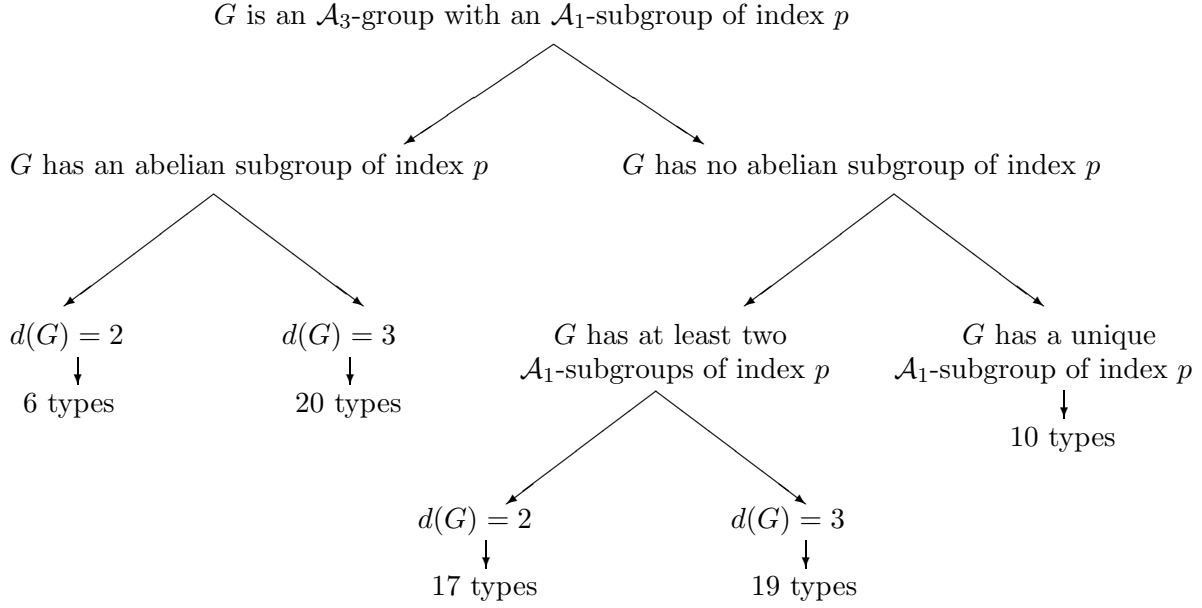
**Problem 893([8]).** *Describe the set  $\{\alpha_1(G) \mid G \text{ is an } \mathcal{A}_3\text{-group}\}$ .*

**Problem 1595([9]).** *Classify the  $\mathcal{A}_3$ -groups  $G$  such that  $\alpha_1(G) < p^2 + p + 1$ , where  $\alpha_1(G)$  denotes the number of  $\mathcal{A}_1$ -subgroups in a  $p$ -group  $G$ .*

**Problem 2829([10]).** *Find  $\max\{\alpha_1(G)\}$ , where a  $p$ -group  $G$  runs over all  $\mathcal{A}_3$ -groups.*

As a corollary of the classification of  $\mathcal{A}_3$ -groups, we get all  $\alpha_1(G)$  for  $\mathcal{A}_3$ -groups  $G$ . Hence these problems mentioned above are also solved. Moreover, we give the triple  $(\mu_0, \mu_1, \mu_2)$ , where  $\mu_i$  denotes the number of  $\mathcal{A}_i$ -subgroups of index  $p$  in  $\mathcal{A}_3$ -groups.

We classify  $\mathcal{A}_3$ -groups  $G$  in two parts:  $G$  has an  $\mathcal{A}_1$ -subgroup of index  $p$ , and  $G$  has no  $\mathcal{A}_1$ -subgroup of index  $p$ . The sketch of the classification of  $\mathcal{A}_3$ -groups are as follows.



## 2 Preliminaries

In this paper,  $p$  is always a prime. We use  $F_p$  to denote the finite field containing  $p$  elements.  $F_p^*$  is the multiplicative group of  $F_p$ .  $(F_p^*)^2 = \{a^2 \mid a \in F_p^*\}$  is a subgroup of  $F_p^*$ .

Let  $G$  be a finite group. We use  $c(G)$ ,  $\exp(G)$  and  $d(G)$  to denote the nilpotency class, the exponent and the minimal number of generators of  $G$  respectively. We use  $C_{p^m}$ ,  $C_{p^m}^n$  and  $H * K$  to denote the cyclic group of order  $p^m$ , the direct product of  $n$  cyclic groups of order  $p^m$ , and a central product of  $H$  and  $K$  respectively. We use  $M \triangleleft G$  to denote  $M$  is a maximal subgroup of  $G$  and

$$G > G' = G_2 > G_3 > \cdots > G_{c+1} = 1$$

denote the lower central series of  $G$ , where  $c = c(G)$ .

Let  $G$  be a finite  $p$ -group. We use  $G \in \mathcal{A}_t$  to denote  $G$  is an  $\mathcal{A}_t$ -group. For any positive integer  $s$ , we define

$$\Omega_s(G) = \langle a \in G \mid a^{p^s} = 1 \rangle \text{ and } \mathcal{U}_s(G) = \langle a^{p^s} \mid a \in G \rangle.$$

We use  $M_p(n, m)$  to denote the  $p$ -groups

$$\langle a, b \mid a^{p^n} = b^{p^m} = 1, a^b = a^{1+p^{n-1}} \rangle, \text{ where } n \geq 2.$$

We use  $M_p(n, m, 1)$  to denote the  $p$ -groups

$$\langle a, b, c \mid a^{p^n} = b^{p^m} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle,$$

where

$$n \geq m, \text{ and if } p = 2, \text{ then } m + n \geq 3.$$

For other notation and terminology the reader is referred to [17].

The following results are used in this paper, we gather them together.

**Lemma 2.1.** ([12, Theorem 2.3]) *Suppose that  $G$  is a finite  $p$ -group. Then  $G$  is metacyclic if and only if  $G/\Phi(G')G_3$  is metacyclic.*

**Lemma 2.2.** ([29, Lemma 2.2]) *Suppose that  $G$  is a finite non-abelian  $p$ -group. Then the following conditions are equivalent:*

- (1)  $G$  is minimal non-abelian;
- (2)  $d(G) = 2$  and  $|G'| = p$ ;
- (3)  $d(G) = 2$  and  $\Phi(G) = Z(G)$ .

**Lemma 2.3.** ([25]) *Suppose that  $G$  is an  $\mathcal{A}_1$ -group. Then  $G$  is one of the following groups:  $Q_8$ ,  $M_p(n, m)$  or  $M_p(n, m, 1)$ .*

**Lemma 2.4.** ([7, §9, Exercise 10]) *Let  $G$  be a 3-group of maximal class. Then the fundamental subgroup  $G_1$  of  $G$  is either abelian or minimal nonabelian.*

**Lemma 2.5.** ([31]) *A finite  $p$ -group  $G$  is an  $\mathcal{A}_2$ -group if and only if  $G$  is one of the following pairwise non-isomorphic groups:*

- (I)  $d(G) = 2$  and  $G$  has an abelian subgroup of index  $p$ . In this case,  $\alpha_1(G) = p$ .
- (1)  $\langle a, b \mid a^8 = b^{2^m} = 1, [a, b] = a^{-2} \rangle$ , where  $m \geq 1$ ;
  - (2)  $\langle a, b \mid a^8 = b^{2^m} = 1, [a, b] = a^2 \rangle$ , where  $m \geq 1$ ;
  - (3)  $\langle a, b \mid a^8 = 1, b^{2^m} = a^4, [a, b] = a^{-2} \rangle$ , where  $m \geq 1$ ;
  - (4)  $\langle a_1, b; a_2, a_3 \mid a_1^p = a_2^p = a_3^p = b^{p^m} = 1, [a_1, b] = a_2, [a_2, b] = a_3, [a_3, b] = 1, [a_i, a_j] = 1 \rangle$ , where  $p \geq 5$  for  $m = 1$ ,  $p \geq 3$  and  $1 \leq i, j \leq 3$ ;
  - (5)  $\langle a_1, b; a_2 \mid a_1^p = a_2^p = b^{p^{m+1}} = 1, [a_1, b] = a_2, [a_2, b] = b^{p^m}, [a_1, a_2] = 1 \rangle$ , where  $p \geq 3$ ;
  - (6)  $\langle a_1, b; a_2 \mid a_1^{p^2} = a_2^p = b^{p^m} = 1, [a_1, b] = a_2, [a_2, b] = a_1^{\nu p}, [a_1, a_2] = 1 \rangle$ , where  $p \geq 3$  and  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ .
  - (7)  $\langle a_1, b; a_2 \mid a_1^9 = a_2^3 = 1, b^3 = a_1^3, [a_1, b] = a_2, [a_2, b] = a_1^{-3}, [a_2, a_1] = 1 \rangle$ .
- (II)  $d(G) = 3$ ,  $|G'| = p$  and  $G$  has an abelian subgroup of index  $p$ . In this case,  $\alpha_1(G) = p^2$ .
- (8)  $\langle a, b, x \mid a^4 = x^2 = 1, b^2 = a^2 = [a, b], [x, a] = [x, b] = 1 \rangle \cong Q_8 \times C_2$ ;
  - (9)  $\langle a, b, x \mid a^{p^{n+1}} = b^{p^m} = x^p = 1, [a, b] = a^{p^n}, [x, a] = [x, b] = 1 \rangle \cong M_p(n+1, m) \times C_p$ ;
  - (10)  $\langle a, b, x; c \mid a^{p^n} = b^{p^m} = c^p = x^p = 1, [a, b] = c, [c, a] = [c, b] = [x, a] = [x, b] = 1 \rangle \cong M_p(n, m, 1) \times C_p$ , where  $n \geq m$ , and  $n \geq 2$  if  $p = 2$ ;
  - (11)  $\langle a, b, x \mid a^4 = 1, b^2 = x^2 = a^2 = [a, b], [x, a] = [x, b] = 1 \rangle \cong Q_8 * C_4$ ;
  - (12)  $\langle a, b, x \mid a^{p^n} = b^{p^m} = x^{p^2} = 1, [a, b] = x^p, [x, a] = [x, b] = 1 \rangle \cong M_p(n, m, 1) * C_{p^2}$ , where  $n \geq 2$  if  $p = 2$  and  $n \geq m$ .
- (III)  $d(G) = 3$ ,  $|G'| = p^2$  and  $G$  has an abelian subgroup of index  $p$ . In this case,  $\alpha_1(G) = p^2 + p$ .
- (13)  $\langle a, b, c \mid a^4 = b^4 = 1, c^2 = a^2 b^2, [a, b] = b^2, [c, a] = a^2, [c, b] = 1 \rangle$ ;
  - (14)  $\langle a, b, d \mid a^{p^m} = b^{p^2} = d^p = 1, [a, b] = a^{p^{m-1}}, [d, a] = b^p, [d, b] = 1 \rangle$ , where  $m \geq 3$  if  $p = 2$ ;
  - (15)  $\langle a, b, d \mid a^{p^m} = b^{p^2} = d^{p^2} = 1, [a, b] = d^p, [d, a] = b^{jp}, [d, b] = 1 \rangle$ , where  $(j, p) = 1$ ,  $p > 2$ ,  $j$  is a fixed quadratic non-residue modulo  $p$ , and  $-4j$  is a quadratic non-residue modulo  $p$ ;

- (16)  $\langle a, b, d \mid a^{p^m} = b^{p^2} = d^{p^2} = 1, [a, b] = d^p, [d, a] = b^{jp}d^p, [d, b] = 1 \rangle$ , where if  $p$  is odd, then  $4j = 1 - \rho^{2r+1}$  with  $1 \leq r \leq \frac{p-1}{2}$  and  $\rho$  the smallest positive integer which is a primitive root (mod  $p$ ); if  $p = 2$ , then  $j = 1$ .
- (IV)  $d(G) = 2$  and  $G$  has no abelian subgroup of index  $p$ . In this case,  $\alpha_1(G) = 1 + p$ .
- (17)  $\langle a, b \mid a^{p^{r+2}} = 1, b^{p^{r+s+t}} = a^{p^{r+s}}, [a, b] = a^{p^r} \rangle$ , where  $r \geq 2$  for  $p = 2$ ,  $r \geq 1$  for  $p \geq 3$ ,  $t \geq 0$ ,  $0 \leq s \leq 2$  and  $r + s \geq 2$ ;
- (18)  $\langle a, b, c \mid a^{p^2} = b^{p^2} = c^p = 1, [a, b] = c, [c, a] = b^{\nu p}, [c, b] = a^p \rangle$ , where  $p \geq 5$ ,  $\nu$  is a fixed quadratic non-residue modulo  $p$ ;
- (19)  $\langle a, b, c \mid a^{p^2} = b^{p^2} = c^p = 1, [a, b] = c, [c, a] = a^{-p}b^{-lp}, [c, b] = a^{-p} \rangle$ , where  $p \geq 5$ ,  $4l = \rho^{2r+1} - 1$ ,  $r = 1, 2, \dots, \frac{1}{2}(p-1)$ ,  $\rho$  is the smallest positive integer which is a primitive root modulo  $p$ ;
- (20)  $\langle a, b, c \mid a^9 = b^9 = c^3 = 1, [a, b] = c, [c, a] = b^{-3}, [c, b] = a^3 \rangle$ ;
- (21)  $\langle a, b, c \mid a^9 = b^9 = c^3 = 1, [a, b] = c, [c, a] = b^{-3}, [c, b] = a^{-3} \rangle$ .
- (V)  $d(G) = 3$  and  $G$  has no abelian subgroup of index  $p$ . In this case,  $\alpha_1(G) = 1 + p + p^2$ .
- (22)  $\langle a, b, d \mid a^4 = b^4 = d^4 = 1, [a, b] = d^2, [d, a] = b^2d^2, [d, b] = a^2b^2, [a^2, b] = [b^2, a] = 1 \rangle$ .

By analyzing the groups in Lemma 2.5, we have following lemma.

**Lemma 2.6.** *Suppose that  $G$  is an  $\mathcal{A}_2$ -group of order  $p^n$ . Then*

- (1)  $d(G) \leq 3$  and  $c(G) \leq 3$ ;
- (2) if  $d(G) = 3$ , then  $c(G) = 2$  and  $G' \leq C_p^3$ . Moreover, if  $G' = C_p^3$ , then  $|G| = 2^6$ ,  $G' = \Omega_1(G) = Z(G) \cong C_2^3$  and  $G$  has no abelian subgroup of index  $p$ ;
- (3) if  $d(G) = 3$  and  $|G'| = p$ , then  $|G : Z(G)| = p^2$  and the type of  $G/G'$  is  $(p^u, p^v, p)$  where  $u + v = n - 2$ ;
- (4) if  $d(G) = 3$  and  $|G'| = p^2$ , then  $c(G) = 2$ ,  $\Phi(G) = \Omega_1(G) = Z(G)$ ,  $G$  has a unique abelian subgroup  $A$  of index  $p$  and  $\exp(A) > p$ ,  $G/G'$  has the type  $(p^{n-4}, p, p)$  and  $A/G'$  has the type  $(p^{n-5}, p, p)$ .
- (5) if  $G$  has an abelian subgroup of index  $p$ , then  $G$  is metacyclic if and only if  $p = 2$ ;
- (6) if  $p = 2$ , then  $G$  is metacyclic if and only if  $d(G) = 2$ ;
- (7) if  $|G'| = p$ , then  $\alpha_1(G) = p^2$ ;
- (8) if  $d(G) = 2$  and  $|G'| = p^3$ , then  $\alpha_1(G) = 1 + p$ ;

(9) if  $d(G) = 2$  and  $G' \cong C_p^2$ , then  $\alpha_1(G) = p$ .

**Lemma 2.7.** ([7, p<sub>27</sub>, Exercise 6]) *Let  $G$  be a non-abelian  $p$ -group. Then the number of abelian subgroups of index  $p$  in  $G$  is  $0, 1$  or  $p + 1$ .*

**Lemma 2.8.** [17, p<sub>259</sub>, Aufgabe 2]) *Suppose that a finite non-abelian  $p$ -group  $G$  has an abelian normal subgroup  $A$ , and  $G/A = \langle bA \rangle$  is cyclic. Then the map  $a \mapsto [a, b]$ ,  $a \in A$ , is an epimorphism from  $A$  to  $G'$ , and  $G' \cong A/A \cap Z(G)$ . In particular, if a non-abelian  $p$ -group  $G$  has an abelian subgroup of index  $p$ , then  $|G| = p|G'| |Z(G)|$ .*

**Lemma 2.9.** *Let  $p$  be an odd prime. Then the equation  $x^2 + ry^2 - u = 0$  about  $x, y$  over  $F_p$  has a solution, and the following conclusions hold:*

(1) *If  $-r \in (F_p^*)^2$  and  $u \in F_p^*$ , then the equation  $x^2 + ry^2 - u = 0$  has exactly  $p - 1$  solutions;*

(2) *If  $-r \notin (F_p^*)^2$  and  $u \in F_p^*$ , then the equation  $x^2 + ry^2 - u = 0$  has exactly  $p + 1$  solutions.*

**Proof** Let  $A = \{a^2 \mid a \in F_p\}$  and  $B = \{u - rb^2 \mid b \in F_p\}$ . Then  $|A| = |B| = \frac{p+1}{2}$ . It follows that  $A \cap B \neq \emptyset$ . Hence there exist  $a, b \in F_p$  such that  $a^2 = u - rb^2$ .

(1) Let  $-r = \alpha^2$ . Then the equation  $x^2 + ry^2 - u = 0$  is  $x^2 - \alpha^2 y^2 = u$ , and hence is equivalent to

$$\begin{cases} x + \alpha y = k \\ x - \alpha y = k^{-1}u \end{cases}$$

where  $k = 1, 2, \dots, p - 1$ . Hence the equation  $x^2 + ry^2 - u = 0$  has exactly  $p - 1$  solutions.

(2) Let  $n(u)$  be the number of solutions of the equation  $x^2 + ry^2 - u = 0$ . Since  $-r \notin (F_p^*)^2$ ,  $n(0) = 1$ . It is obvious that  $n(u_1) = n(u_2)$  for  $u_1 u_2^{-1} \in (F_p^*)^2$ . Let  $\nu \notin (F_p^*)^2$ . Since  $F_p \times F_p$  has a partition

$$\{(x, y) \in F_p \times F_p \mid x^2 + ry^2 \in (F_p^*)^2\} \cup \{(x, y) \in F_p \times F_p \mid x^2 + ry^2 \notin (F_p^*)^2\},$$

$p^2 = n(0) + \frac{p-1}{2}n(1) + \frac{p-1}{2}n(\nu)$ . It follows that  $n(1) + n(\nu) = 2p + 2$ . If we prove that  $n(1) = p + 1$ , then  $n(u) = p + 1$  for all  $u \in F_p^*$ .

Now we calculate  $n(1)$ . If  $y = 0$ , then the equation  $x^2 + ry^2 - 1 = 0$  has two solutions  $(1, 0)$  and  $(-1, 0)$ . If  $y \neq 0$ , then the equation  $x^2 + ry^2 - 1 = 0$  is equivalent to  $(xy^{-1})^2 - y^{-2} + r = 0$ . By (1), the later has  $p - 1$  solutions. Hence  $n(1) = p + 1$ .  $\square$

**Lemma 2.10.** *Let  $p$  be an odd prime. If  $s^2 - 4r \not\equiv 0 \pmod{p}$ , then the equation  $x^2 + sxy + ry^2 + wx + vy + u = 0$  about  $x, y$  over  $F_p$  has a solution.*

**Proof** Let  $x_1 = x + 2^{-1}sy$ ,  $y_1 = y$ ,  $r_1 = r - 4^{-1}s^2$  and  $v_1 = v - 2^{-1}ws$ . Then the equation is turned to  $x_1^2 + r_1 y_1^2 + w x_1 + v_1 y_1 + u = 0$ . Let  $x_2 = x_1 + 2^{-1}w$ ,  $y_2 = y_1 + 2^{-1}r_1^{-1}v_1$  and  $u_2 = u - 4^{-1}w^2 - 4^{-1}r_1^{-1}v_1^2$ . Then the equation is turned to  $x_2^2 + r_1 y_2^2 + u_2 = 0$ . By Lemma 2.9, the last equation has a solution  $(a, b)$ . Thus  $(a - 2^{-1}w - 2^{-1}sb + 4^{-1}sr_1^{-1}v_1, b - 2^{-1}r_1^{-1}v_1)$  is a solution of the first equation.  $\square$

**Lemma 2.11.** ([29, Lemma 3.1]) *Let  $G$  be a non-abelian two-generator  $p$ -group having an abelian subgroup  $A$  of index  $p$ . Assume that  $|G/G'| = p^{m+1}$  and  $c(G) = c$ . Then  $m \geq 1$ ,  $c \geq 2$  and*

(1)  *$G$  has the lower central complexion  $(m+1, \underbrace{1, \dots, 1}_{c-1})$  and hence  $|G'| = p^{c-1}$ ,  $|G| = p^{m+c}$ ;*

(2)  *$|Z(G)| = p^m$  and  $G/Z(G)$  is of maximal class;*

(3)  *$Z(G) \leq \Phi(G)$ ,  $\Phi(G) = G'Z(G)$  and  $G' \cap Z(G) = G_c$ ;*

(4) *Let  $M$  be a non-abelian subgroup of index  $p$  of  $G$ . Then  $Z(M) = Z(G)$  and*

$$M' = G_3, M_3 = G_4, \dots, M_{c-1} = G_c.$$

**Lemma 2.12.** ([29, Corollary 3.8]) *Suppose that  $G$  is a finite  $p$ -group having an abelian subgroup of index  $p$ , and all non-abelian subgroups of  $G$  are generated by two elements. Then*

(1)  *$G \in \mathcal{A}_2$  if and only if  $c(G) = 3$ ;*

(2) *If  $p = 2$ , then  $G$  is metacyclic;*

(3) *If  $c(G) \leq p$ , then  $d(G') = c(G) - 1$ . If  $c(G) \geq p + 1$ , then  $d(G') = p - 1$ .*

**Lemma 2.13.** ([29, Theorem 4.1]) *Suppose that  $G$  is a three-generator non-abelian  $p$ -group, and all non-abelian proper subgroups of  $G$  are generated by two elements. If  $G$  has an abelian subgroup of index  $p$ , then  $G \in \mathcal{A}_2$ .*

**Lemma 2.14.** ([13, Theorem 4]) *Let  $G$  be a  $p$ -group. If both  $G$  and  $G'$  can be generated by two elements, then  $G'$  is abelian.*

**Lemma 2.15.** *Suppose that  $G$  is a finite  $p$ -group.  $M_1$  and  $M_2$  are two distinct maximal subgroups of  $G$ . Then  $|G'| \leq p|M'_1M'_2|$ .*

**Proof** Let  $\bar{G} = G/M'_1M'_2$ . Then  $\bar{G}$  has abelian subgroups  $\bar{M}_1$  and  $\bar{M}_2$  of index  $p$ . Hence  $Z(\bar{G}) \geq \bar{M}_1 \cap \bar{M}_2$ . By Lemma 2.8,  $|\bar{G}'| \leq p$ . Thus  $|G'| \leq p|M'_1M'_2|$ .  $\square$

**Lemma 2.16.** ([1, Theorem 5.6]) *If  $G$  is a finite non-abelian  $p$ -group and  $|G'| = p^k$ , then  $G$  has a subgroup  $K$  such that  $d(K) \leq k + 1$  and  $K_n = G_n$  for all  $2 \leq n \leq c(G)$ .*

**Proposition 2.17.** ([27]) *Let  $G$  be a metabelian group and  $a, b \in G$ . For any positive integers  $i$  and  $j$ , let*

$$[ia, jb] = [a, b, \underbrace{a, \dots, a}_{i-1}, \underbrace{b, \dots, b}_{j-1}].$$

*Then, for any positive integers  $m$  and  $n$ ,*

$$[a^m, b^n] = \prod_{i=1}^m \prod_{j=1}^n [ia, jb]^{\binom{m}{i} \binom{n}{j}}.$$



The following lemma is equivalent to [19, Theorem 3.4]

**Lemma 2.18.** *Assume  $G$  is a finite  $p$ -group. If  $c(H) \leq 2$  for all  $H < G$ , then  $c(G) \leq 3$ .*

Let  $G$  be a group of order  $p^m$ ,  $|G : \Phi(G)| = p^d$ ,

$$\Gamma_i = \{H < G \mid \Phi(G) \leq H, |G : H| = p^i\}$$

and  $\mathfrak{M}$  be a set of proper subgroups of  $G$ . For  $K \leq G$ , we denote by  $\alpha(K)$  the number of members of the set  $\mathfrak{M}$  that are subgroups of  $K$ . Obviously,  $\alpha(G) = |\mathfrak{M}|$ .

**Theorem 2.19.** (*Hall's enumeration principle*) *In the above notation,*

$$\alpha(G) = \sum_{i=1}^d \sum_{H \in \Gamma_i} (-1)^{i-1} p^{\binom{i}{2}} \alpha(H).$$

### 3 Some general properties of $\mathcal{A}_t$ -groups with $t \geq 3$

Y. Berkovich was the first where  $\mathcal{A}_t$ -groups were investigated. He obtained many significant results. The following are some related to  $\mathcal{A}_3$ -groups.

**Lemma 3.1.** ([5, Lemma J(i)]) *Let  $G$  be a metacyclic  $p$ -group. Then  $G$  is an  $\mathcal{A}_t$ -group if and only if  $|G'| = p^t$ .*

**Lemma 3.2.** ([8, Proposition 72.2]) *If a  $p$ -group  $G$  is an  $\mathcal{A}_3$ -group, then  $|G'| \leq p^4$ . For  $p = 5$ , there exists an  $\mathcal{A}_3$ -group  $G$  with  $|G'| = 5^4$ .*

**Lemma 3.3.** ([8, Proposition 72.3]) *Suppose that a  $p$ -group  $G$  is an  $\mathcal{A}_3$ -group with  $|G'| = p^4$ . Then*

- (a)  $G'$  is abelian.
- (b) If  $\exp(G') = p$ , then  $p > 2$ ,  $G/G'$  is abelian of type  $(p^n, p)$  ( $n \geq 1$ ) and  $G' \cong C_p^m$ .  
If, in addition,  $n > 1$ , then  $\Omega_1(G) = G'$ .
- (c)  $G'$  is not metacyclic.

**Theorem 3.4.** ([8, Theorem 72.6]) *If a  $p$ -group  $G$  is an  $\mathcal{A}_4$ -group, then  $|G'| \leq p^6$ .*

In following we give more information about  $\mathcal{A}_t$ -groups.

**Lemma 3.5.** ([2, Corollary 2.4]) (1) *Let  $M$  be an  $\mathcal{A}_t$ -group, and  $A$  be an abelian group of order  $p^k$ . Then  $G = M \times A$  is an  $\mathcal{A}_{t+k}$ -group.*

(2) *Let  $M$  be an  $\mathcal{A}_t$ -group with  $|M'| = p$ ,  $G = M * A$ , where  $A$  is an abelian group of order  $p^{k+1}$  and  $M \cap A = G'$ . Then  $G$  is an  $\mathcal{A}_{t+k}$ -group.*

**Lemma 3.6.** ([24, Theorem 5.4]) *Assume  $G$  is a finite 2-group having a unique  $\mathcal{A}_1$ -subgroup  $A$  of index  $p$ . If  $|G| = 2^t \geq 2^9$  and  $G/A'$  has a unique abelian subgroup of index  $p$ , then one of the following is true:*

- (1)  $G$  is metacyclic;
- (2)  $G$  is an  $\mathcal{A}_{t-2}$ -group or an  $\mathcal{A}_{t-3}$ -group.

**Lemma 3.7.** ([24, Corollary 5.5]) *Assume that  $G$  is a finite 2-group having a unique  $\mathcal{A}_1$ -subgroup  $A$  of index  $p$ . If  $G$  is an  $\mathcal{A}_3$ -group and  $|G| \geq 2^9$ , then  $G$  is metacyclic.*

**Lemma 3.8.** *Suppose that  $G$  is a finite  $p$ -group having an abelian subgroup of index  $p$ . If all non-abelian subgroups of  $G$  are generated by two elements, then  $G$  is an  $\mathcal{A}_t$ -group if and only if  $c(G) = t + 1$ .*

**Proof** Suppose that  $c(G) = t + 1$  and  $M$  is a non-abelian subgroup of index  $p$ . By Lemma 2.11(4),  $c(M) = t$ . By hypothesis,  $d(M) = 2$ . By induction on  $t$ ,  $M$  is an  $\mathcal{A}_{t-1}$ -group. Hence  $G$  is an  $\mathcal{A}_t$ -group.  $\square$

**Lemma 3.9.** *If  $G$  is a  $p$ -group of maximal class of order  $p^n$  with  $n \geq 3$ , then  $G$  is an  $\mathcal{A}_{n-2}$ -group.*

**Proof** By [7, Theorem 9.6(f)], there exists a subgroup  $A$  of order  $p^2$  of  $G$  such that  $C_G(A) = A$ . Let  $B$  be a subgroup of  $G$  with  $B \geq A$  and  $|B| = p^3$ . Then  $B$  is not abelian. It follows that  $G \in \mathcal{A}_{n-2}$ .  $\square$

**Lemma 3.10.** *If a  $p$ -group  $G$  is an  $\mathcal{A}_t$ -group and  $G'$  is regular, then  $\exp(G') \leq p^t$ .*

**Proof** By induction on  $t$  we have  $\exp(M') \leq p^{t-1}$  for all  $M \triangleleft G$ . Let  $N = \prod_{M \triangleleft G} M'$ . Since  $G'$  is regular,  $\exp(N) \leq p^{t-1}$  by [7, Theorem 7.2(b)]. It is easy to see that  $G/N$  is abelian or an  $\mathcal{A}_1$ -group. It follows by Lemma 2.2(2) that  $|G'N/N| = |(G/N)'| \leq p$ . Thus  $\exp(G') \leq p \cdot p^{t-1} = p^t$ .  $\square$

**Corollary 3.11.** *Assume  $G$  is a finite  $p$ -group.*

- (1) *If  $G$  is an  $\mathcal{A}_3$ -group, then  $\exp(G') \leq p^3$ .*
- (2) *If  $p \geq 7$  and  $G$  is an  $\mathcal{A}_4$ -group, then  $\exp(G') \leq p^4$ .*

**Proof** By Lemma 3.3(a), Theorem 3.4 and [7, Theorem 7.1(b)] we get  $G'$  is regular. Thus the conclusion follows by Lemma 3.10.  $\square$

**Theorem 3.12.** *If a  $p$ -group  $G$  is an  $\mathcal{A}_3$ -group, then  $c(G) \leq 4$ .*

**Proof** Let  $N = \prod_{M \triangleleft G} M_3$ . Since  $M_3$  is characteristic in  $M$ ,  $M_3 \triangleleft G$ . Thus  $N \triangleleft G$ . By Lemma 2.18,  $c(G/N) \leq 3$ . If  $N \leq Z(G)$ , then  $c(G) \leq 4$ . Assume  $N \not\leq Z(G)$ . Then there exists  $M \triangleleft G$  such that  $M_3 \not\leq Z(G)$ . Thus  $|M_3| \geq p^2$ . Notice that  $M$  is an  $\mathcal{A}_2$ -group. Hence  $|M| = p^5$  by checking the list of groups in Lemma 2.5. Thus  $|G| = p^6$ . If  $c(G) = 5$ , then  $G$  is of maximal class. It follows by Lemma 3.9 that  $G$  is an  $\mathcal{A}_4$ -group. This is a contradiction.  $\square$

**Remark 3.13.** *There exist  $\mathcal{A}_3$ -groups of class 4. In fact, those groups listed in Theorem 5.1 are all  $\mathcal{A}_3$ -groups of class 4.*

**Lemma 3.14.** *Assume a  $p$ -group  $G$  is an  $\mathcal{A}_3$ -group.*

- (1) *If  $\Phi(G)$  is non-abelian, then  $\Phi(G)$  is metacyclic.*
- (2) *If  $p > 2$ , then  $\Phi(G)$  is non-abelian if and only if  $G$  is metacyclic.*

**Proof** (1) Since  $G$  is an  $\mathcal{A}_3$ -group,  $d(G) = 2$  and  $\Phi(G) \in \mathcal{A}_1$ . By Lemma 2.2,  $d(\Phi(G)) = 2$ . It follows from [7, Theorem 44.12] that  $\Phi(G)$  is metacyclic.

(2)  $\Leftarrow$  Let  $G = \langle a, b \rangle$ . Then  $o([a, b]) = p^3$  by Lemma 3.1. Since  $G$  is regular,  $[a^p, b^p] = 1$  if and only if  $[a, b]^{p^2} = 1$  by [7, Theorem 7.2(e)]. Since  $o([a, b]) = p^3$ ,  $[a^p, b^p] \neq 1$ . It follows that  $\Phi(G)$  is non-abelian.

$\Rightarrow$  Assume  $G$  is not metacyclic. It follows by Lemma 3.1 and Lemma 3.3(a) and (c) that  $|G' : \Omega_1(G')| \leq p$ . Thus  $G_3 \leq \Omega_1(G')$ . It follows that  $\exp(G_3) \leq p$ . on the other hand,  $c(G) \leq 4$  by Lemma 3.12. Since  $G'$  is abelian,  $G$  is metabelian. By lemma 2.17 we have

$$[x^p, y^p] = \prod_{i=1}^p \prod_{j=1}^p [ix, jy]^{(p)_i (p)_j}, \quad [x^p, z] = \prod_{i=1}^p [ix, z]^{(p)_i}.$$

where  $x, y \in G, z \in G'$ .

It follows from  $\exp(G') \leq p^2$ ,  $\exp(G_3) \leq p$ ,  $c(G) \leq 4$  and  $p > 2$  that  $[x^p, y^p] = 1$  and  $[x^p, z] = 1$ . Thus  $\Phi(G)$  is abelian. This is a contradiction.  $\square$

**Remark 3.15.** *In Lemma 3.14, if  $p = 2$ , then (2) is not true. For example,  $D_{32}$  is a metacyclic  $\mathcal{A}_3$ -group, but  $\Phi(D_{32}) \cong C_8$ .*

**Lemma 3.16.** *Suppose that  $G$  is an  $\mathcal{A}_3$ -group. If  $G$  has a maximal subgroup  $M$  such that  $d(M) = 3$  and  $M' \not\leq Z(G)$ , then  $G$  has a three-generator maximal subgroup  $L$  such that  $d(L) = 3$ ,  $|L'| \leq p^2$  and  $L' \not\leq Z(G)$ .*

**Proof** Otherwise, for every three-generator maximal subgroup  $L$  with  $d(L) = 3$  and  $|L'| \leq p^2$ , we have  $L' \leq Z(G)$ . Hence  $|M'| \geq p^3$ . It follows from Lemma 2.6(2) that

$$p = 2, |M| = 2^6 \text{ and } M' = \Omega_1(M) = Z(M) \cong C_2^3.$$

Let  $d(G) = r$  where  $2 \leq r \leq 4$ . Then  $G$  has  $s = 1 + 2 + \cdots + 2^{r-1}$  maximal subgroups. Let  $H_1, H_2, \dots, H_{s-1}$  and  $M$  be all maximal subgroups of  $G$ . Since  $M' \leq H_i$ ,  $H_i$  is not metacyclic. It follows from Lemma 2.6(6) that  $d(H_i) = 3$ . If  $|H'| = 8$ , then we also have  $H' = \Omega_1(H) = Z(H)$ . Since  $M' \cong C_2^3 \leq H$ ,  $M' = H'$ . It follows that  $[M', G] = [M', HM] = 1$  and hence  $M' \leq Z(G)$ , a contradiction. Hence  $|H'_i| \leq 4$  and  $H'_i \leq Z(G)$ . Let  $N = H'_1 H'_2 \cdots H'_{s-1}$ . Then  $G/N$  has  $s - 1$  abelian subgroups of index  $p$ . Since  $s - 1 \geq 2$ , we have, by Lemma 2.7, the number of abelian subgroups of index  $p$  of  $G/N$  is 3. Hence  $s - 1 \leq 3$ . It follows that  $d(G) = 2$  and  $M/N$  is also abelian. Thus  $M' \leq N \leq Z(G)$ , a contradiction.  $\square$

**Lemma 3.17.** *Suppose that  $G$  is a four-generator  $\mathcal{A}_3$ -group. Then  $c(G) = 2$ ,  $\Phi(G) \leq Z(G)$ ,  $G' \leq C_p^3$  and all  $\mathcal{A}_1$ -subgroups of  $G$  contain  $\Phi(G)$ .*

**Proof** Let  $M$  be an  $\mathcal{A}_1$ -subgroup of  $G$  and  $L$  be a maximal subgroup containing  $M$ . Since  $d(G) = 4$ , we have  $d(L) \geq 3$ . Hence  $L \in \mathcal{A}_2$ . By Lemma 2.6(2), we get  $d(L) = 3$ ,  $c(L) = 2$  and  $L' \leq C_p^3$ . By comparing the indexes of  $\Phi(M)$ ,  $\Phi(L)$  and  $\Phi(G)$ , we get

$$Z(M) = \Phi(M) = \Phi(L) = \Phi(G).$$

By the arbitrariness of  $M$  we get  $\Phi(G) \leq Z(G)$ . Hence  $c(G) = 2$ . Let  $N = \prod_{L < G} L'$ . Then  $N \leq Z(G)$  and  $\exp(N) = p$ . Since all subgroups of  $G/N$  are abelian, we have  $G/N$  is also abelian. It follows that  $G' = N$ . Hence  $\exp(G') = p$ . Since  $G' \leq \Omega_1(\Phi(G)) = \Omega_1(\Phi(M))$ ,  $G' \leq C_p^3$ .  $\square$

**Lemma 3.18.** *Let  $G$  be an  $\mathcal{A}_3$ -group such that  $d(G) = 3$  and  $\Phi(G) \leq Z(G)$ . Then  $\alpha_1(G) = \mu_1 + p^2\mu_2$ .*

**Proof** Groups satisfying  $d(G) = 3$  and  $\Phi(G) \leq Z(G)$  were classified in [2, 23]. The derived groups of non-abelian maximal subgroups of  $G$  are of order  $p$ .

Let  $H \in \Gamma_2$ . Then, by  $\Phi(G) \leq Z(G)$ ,  $H$  is abelian and hence  $\alpha_1(H) = 0$ . Let  $M \in \Gamma_1$ . If  $M \in \mathcal{A}_2$ , then, by Lemma 2.6(7),  $\alpha_1(M) = p^2$ . By Hall's enumeration principle,  $\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = \mu_1 + p^2\mu_2$ .  $\square$

## 4 $\mathcal{A}_3$ -groups with an $\mathcal{A}_1$ -subgroup of index $p$

In this section, we classify  $\mathcal{A}_3$ -groups with an  $\mathcal{A}_1$ -subgroup of index  $p$ . Since finite  $p$ -groups with an  $\mathcal{A}_1$ -subgroup of index  $p$  are classified by [2, 3, 22, 23, 24], it is enough to check whether those groups in [2, 3, 22, 23, 24] are  $\mathcal{A}_3$ -groups or not. In this case,  $\mathcal{A}_3$ -groups have 72 non-isomorphic types.

For convenience, in this section, assume  $G$  is an  $\mathcal{A}_3$ -group with an  $\mathcal{A}_1$ -subgroup of index  $p$  in Theorem 4.1, 4.2, 4.4, 4.6 and 4.7. It is easy to see that  $d(G) \leq 3$ .

**Theorem 4.1.**  *$G$  has an abelian subgroup of index  $p$  and  $d(G) = 2$  if and only if  $G$  is isomorphic to one of the following pairwise non-isomorphic groups:*

(Ai)  $c(G) = 3$  and  $G' \cong C_4$ .

(A1)  $\langle a, b; c \mid a^4 = b^2 = c^4 = 1, [a, b] = c, [c, a] = [c, b] = c^2 \rangle$ . Moreover  $|G| = 2^5$ ,  $\Phi(G) = \langle a^2, c \rangle \cong C_4 \times C_2$ ,  $G' = \langle c \rangle$  and  $Z(G) = \langle a^2, c^2 \rangle \cong C_2^2$ .

(A2)  $\langle a, b; c \mid a^4 = b^4 = 1, c^2 = a^2, [a, b] = c, [c, a] = [c, b] = c^2 \rangle$ . Moreover,  $|G| = 2^5$ ,  $\Phi(G) = \langle b^2, c \rangle \cong C_4 \times C_2$ ,  $G' = \langle c \rangle$  and  $Z(G) = \langle b^2, c^2 \rangle \cong C_2^2$ .

(A3)  $\langle a, b; c \mid a^8 = b^2 = 1, c^2 = a^4, [a, b] = c, [c, a] = [c, b] = c^2 \rangle$ . Moreover,  $|G| = 2^5$ ,  $\Phi(G) = \langle a^2, c \rangle \cong C_4 \times C_2$ ,  $G' = \langle c \rangle$  and  $Z(G) = \langle a^2 \rangle \cong C_4$ .

(Aii)  $c(G) = 3$  and  $G' \cong C_p^2$  where  $p > 2$ .

- (A4)  $\langle a, b; c \mid a^{p^3} = b^p = c^p = 1, [a, b] = c, [c, a] = 1, [c, b] = a^{\nu p^2} \rangle$ , where  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ . Moreover,  $|G| = p^5$ ,  $\Phi(G) = \langle a^p, c \rangle \cong C_{p^2} \times C_p$ ,  $G' = \langle a^{p^2}, c \rangle$  and  $Z(G) = \langle a^p \rangle \cong C_{p^2}$ .
- (A5)  $\langle a, b; c \mid a^{p^2} = b^{p^2} = c^p = 1, [a, b] = c, [c, a] = 1, [c, b] = b^p \rangle$ . Moreover,  $|G| = p^5$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_p^3$ ,  $G' = \langle b^p, c \rangle$  and  $Z(G) = \langle a^p, b^p \rangle \cong C_p^2$ .
- (A6)  $\langle a, b; c, d \mid a^{p^2} = b^p = c^p = d^p = 1, [a, b] = c, [c, a] = 1, [c, b] = d, [d, a] = [d, b] = 1 \rangle$ . Moreover,  $|G| = p^5$ ,  $\Phi(G) = \langle a^p, c, d \rangle \cong C_p^3$ ,  $G' = \langle c, d \rangle$  and  $Z(G) = \langle a^p, d \rangle \cong C_p^2$ .

Moreover,  $(\mu_0, \mu_1, \mu_2) = (1, p-1, 1)$  and  $\alpha_1(G) = p^2 + p - 1$ .

**Proof** Let  $A$  be an abelian subgroup of index  $p$  and  $B$  be an  $\mathcal{A}_1$ -subgroup of index  $p$ . By Lemma 2.11,  $G_3 = B'$ ,  $c(G) = 3$  and  $|G'| = p^2$ . Let  $\bar{G} = G/G_3$ . Then  $|\bar{G}'| = p$ . Since  $d(\bar{G}) = d(G) = 2$ ,  $\bar{G} \in \mathcal{A}_1$  by Lemma 2.2. Obviously,  $G_3 = \Phi(G')G_3$ . If  $\bar{G}$  is metacyclic, then  $G$  is also metacyclic by Lemma 2.1. Hence  $G$  is an  $\mathcal{A}_2$ -group by Lemma 3.1. This contradicts that  $G$  is an  $\mathcal{A}_3$ -group. Thus  $\bar{G}$  is a non-metacyclic  $\mathcal{A}_1$ -group. By Lemma 2.3,  $\bar{G} \cong M_p(n, m, 1)$ . Now we have  $\Phi(G')G_3 \leq C_p^2$ ,  $\Phi(G')G_3 \leq Z(G)$  and  $G/\Phi(G')G_3 \cong M_p(n, m, 1)$ . Thus  $G$  is one of the groups classified in [3]. Since  $|G'| = p^2$ ,  $G$  is either one of the groups listed in [3, Theorem 3.5] or [3, Theorem 4.6]. By hypothesis, we only need pick out those groups satisfying the following three conditions:

- (1) the minimal index of  $\mathcal{A}_1$ -subgroups is 1;
- (2) the maximal index of  $\mathcal{A}_1$ -subgroups is 2;
- (3)  $G$  has an abelian subgroup of index  $p$ .

If  $G' \cong C_{p^2}$ , then  $G$  is one of the groups listed in [3, Theorem 3.5]. [3, Theorem 3.1 & 3.6] tell us those groups in [3, Theorem 3.5] satisfy the conditions (1) and (2). Moreover, it is easy to check which one satisfies the condition (3) in these groups. Thus we get the groups (A1)–(A3). If  $G' \cong C_p^2$ , then  $G$  is one of the groups listed in [3, Theorem 4.6]. [3, Theorem 4.1 & 4.7] tell us those groups in [3, Theorem 4.6] satisfy the conditions (1) and (2). Moreover, in these groups we check which one satisfies the condition (3). Thus we get the groups (A4)–(A6). For convenience, in Table 1 we give the relationship between the groups in Theorem 4.1 and those groups in paper [3].

Groups in Theorem 4.1	Groups in [3, Theorem 3.5]	Groups in Theorem 4.1	Groups in [3, Theorem 4.6]
(A1)	(A1)	(A4)	(J1) where $n = 2$ and $m = 1$
(A2)	(A2)	(A5)	(J3) where $n = 2$ and $m = 1$
(A3)	(A3)	(A6)	(J5) where $n = 2$ and $m = 1$

Table 1: The correspondence from Theorem 4.1 to [3, Theorem 3.5 & 4.6]

In following, we calculate the  $(\mu_0, \mu_1, \mu_2)$  and  $\alpha_1(G)$ .

Since  $d(G) = 2$ ,  $G$  has  $1 + p$  maximal subgroups. For any  $\mathcal{A}_3$ -group, we always have  $\mu_2 \geq 1$ . By the hypothesis,  $\mu_0 \geq 1$  and  $\mu_1 \geq 1$ .

If  $G$  is one of the groups (A1)–(A3), then  $p = 2$ . Hence  $\mu_0 + \mu_1 + \mu_2 = 1 + p = 3$ . It follows that  $(\mu_0, \mu_1, \mu_2) = (1, 1, 1)$ .

If  $G$  is one of the groups (A4)–(A6), then it is easy to see that  $\langle c, a, \Phi(G) \rangle$  and  $\langle c, b, \Phi(G) \rangle$  are the unique abelian subgroup of index  $p$  and the  $\mathcal{A}_2$ -subgroup of index  $p$  respectively. Hence  $(\mu_0, \mu_1, \mu_2) = (1, p-1, 1)$ .

Now we calculate  $\alpha_1(G)$ . It is obvious that  $\alpha_1(\Phi(G)) = 0$ . Let  $M \in \Gamma_1$ . If  $M$  is the  $\mathcal{A}_2$ -subgroup of index  $p$ , then  $|M'| = p$ . By Lemma 2.6 (7),  $\alpha_1(M) = p^2$ .

By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) - p\alpha_1(\Phi(G)) = \sum_{H \in \Gamma_1} \alpha_1(H) = \mu_1 + p^2\mu_2 = p^2 + p - 1.$$

□

**Theorem 4.2.**  $G$  has an abelian subgroup of index  $p$  and  $d(G) = 3$  if and only if  $G$  is isomorphic to one of the following pairwise non-isomorphic groups:

(Bi)  $c(G) = 2$  and  $G' \cong C_p$ .

(B1)  $\langle a, b, c \mid a^4 = c^4 = 1, b^2 = a^2 = [a, b], [c, a] = [c, b] = 1 \rangle \cong Q_8 \times C_4$ ; where  $|G| = 2^5$ ,  $\Phi(G) = \langle a^2, c^2 \rangle \cong C_2 \times C_2$ ,  $G' = \langle a^2 \rangle$  and  $Z(G) = \langle a^2, c \rangle \cong C_4 \times C_2$ .

(B2)  $\langle a, b, c \mid a^{p^{n+1}} = b^{p^m} = c^{p^2} = 1, [a, b] = a^{p^n}, [c, a] = [c, b] = 1 \rangle \cong M_p(n+1, m) \times C_{p^2}$ , where  $\min\{n, m\} = 1$ ; where  $|G| = p^{m+n+3}$ ,  $\Phi(G) = \langle a^p, b^p, c^p \rangle \cong C_{p^n} \times C_{p^{m-1}} \times C_p$  if  $m > 1$ ,  $\Phi(G) \cong C_{p^n} \times C_p$  if  $m = 1$ ,  $G' = \langle a^{p^n} \rangle$ ,  $Z(G) = \langle a^p, b^p, c \rangle \cong C_{p^n} \times C_{p^{m-1}} \times C_{p^2}$  if  $m > 1$  and  $Z(G) \cong C_{p^n} \times C_{p^2}$  if  $m = 1$ .

(B3)  $\langle a, b, c, d \mid a^{p^n} = b^p = c^{p^2} = d^p = 1, [a, b] = d, [c, a] = [c, b] = 1 \rangle \cong M_p(n, 1, 1) \times C_{p^2}$ , where  $n \geq 2$  for  $p = 2$ ; where  $|G| = p^{n+4}$ ,  $\Phi(G) = \langle a^p, c^p, d \rangle \cong C_{p^{n-1}} \times C_p \times C_p$  if  $n > 1$ ,  $\Phi(G) \cong C_p \times C_p$  if  $n = 1$ ,  $G' = \langle d \rangle$ ,  $Z(G) = \langle a^p, c, d \rangle \cong C_{p^{n-1}} \times C_{p^2} \times C_p$  if  $n > 1$ ,  $Z(G) \cong C_{p^2} \times C_p$  if  $n = 1$ .

(B4)  $\langle a, b, c \mid a^4 = 1, b^2 = c^4 = a^2 = [a, b], [c, a] = [c, b] = 1 \rangle \cong Q_8 * C_8$ ; where  $|G| = 2^5$ ,  $\Phi(G) = \langle c^2 \rangle \cong C_4$ ,  $G' = \langle c^4 \rangle$  and  $Z(G) = \langle c \rangle \cong C_8$ .

(B5)  $\langle a, b, c \mid a^{p^n} = b^p = c^{p^3} = 1, [a, b] = c^{p^2}, [c, a] = [c, b] = 1 \rangle \cong M_p(n, 1, 1) * C_{p^3}$ , where  $n \geq 2$  for  $p = 2$ . Moreover,  $|G| = p^{n+4}$ ,  $\Phi(G) = \langle a^p, c^p \rangle \cong C_{p^{n-1}} \times C_{p^2}$  if  $n > 1$ ,  $\Phi(G) \cong C_{p^2}$  if  $n = 1$ ,  $G' = \langle c^{p^2} \rangle$ ,  $Z(G) = \langle a^p, c \rangle \cong C_{p^{n-1}} \times C_{p^3}$  if  $n > 1$ ,  $Z(G) \cong C_{p^3}$  if  $n = 1$ .

(Bii)  $c(G) = 2$  and  $G' \cong C_p^2$ .

(B6)  $\langle a, b, c \mid a^p = b^{p^2} = c^{p^2} = 1, [b, c] = 1, [c, a] = c^p, [a, b] = b^{-p} \rangle$ , where  $p$  is odd; and  $|G| = p^5$ ,  $\Phi(G) = G' = Z(G) = \langle b^p, c^p \rangle \cong C_p^2$ .

- (B7)  $\langle a, b, c \mid a^{p^l} = b^{p^2} = c^{p^2} = 1, [b, c] = 1, [c, a] = b^p c^p, [a, b] = b^{-p} \rangle$ , where  $p$  is odd; and  $|G| = p^{l+4}$ ,  $\Phi(G) = Z(G) = \langle a^p, b^p, c^p \rangle \cong C_{p^{l-1}} \times C_p \times C_p$  if  $l > 1$ ,  $\Phi(G) = Z(G) \cong C_p \times C_p$  if  $l = 1$ ,  $G' = \langle b^p, c^p \rangle$ .
- (B8)  $\langle a, b, c \mid a^{p^l} = b^{p^2} = c^{p^2} = 1, [b, c] = 1, [c, a] = b^p c^{tp}, [a, b] = b^{-tp} c^{\nu p} \rangle$ , where  $p$  is odd,  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ ,  $-\nu \in (F_p^*)^2$ , and  $t \in \{0, 1, \dots, \frac{p-1}{2}\}$  such that  $t^2 \neq -\nu$ ; and  $|G| = p^{l+4}$ ,  $\Phi(G) = Z(G) = \langle a^p, b^p, c^p \rangle \cong C_{p^{l-1}} \times C_p \times C_p$  if  $l > 1$ ,  $\Phi(G) = Z(G) \cong C_p \times C_p$  if  $l = 1$ ,  $G' = \langle b^p, c^p \rangle$ .
- (B9)  $\langle a, b, c \mid a^p = b^{p^3} = c^{p^3} = 1, [b, c] = 1, [c, a] = b^{p^2} c^{tp^2}, [a, b] = b^{-tp^2} c^{\nu p^2} \rangle$ , where  $p$  is odd,  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ ,  $-\nu \notin (F_p^*)^2$ , and  $t \in \{0, 1, \dots, \frac{p-1}{2}\}$ ; moreover,  $|G| = p^7$ ,  $\Phi(G) = Z(G) = \langle b^p, c^p \rangle \cong C_{p^2} \times C_{p^2}$ ,  $G' = \langle b^{p^2}, c^{p^2} \rangle$ .
- (B10)  $\langle a, b, c \mid a^{2^l} = b^4 = c^4 = 1, [b, c] = 1, [c, a] = b^2, [a, b] = c^2 \rangle$ ; where  $|G| = 2^{l+4}$ ,  $\Phi(G) = Z(G) = \langle a^2, b^2, c^2 \rangle \cong C_{2^{l-1}} \times C_2 \times C_2$  if  $l > 1$ ,  $\Phi(G) = Z(G) \cong C_2 \times C_2$  if  $l = 1$ ,  $G' = \langle b^2, c^2 \rangle$ .
- (B11)  $\langle a, b, c \mid a^2 = b^8 = c^8 = 1, [b, c] = 1, [c, a] = b^4, [a, b] = b^4 c^4 \rangle$ ; where  $|G| = 2^7$ ,  $\Phi(G) = Z(G) = \langle b^2, c^2 \rangle \cong C_4 \times C_4$ ,  $G' = \langle b^4, c^4 \rangle$ .
- (B12)  $\langle a, b, c \mid a^{p^l} = b^{p^3} = c^{p^2} = 1, [b, c] = 1, [a, b] = c^{\nu p}, [c, a] = b^{p^2} \rangle$ , where  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ ; and  $|G| = p^{l+5}$ ,  $\Phi(G) = Z(G) = \langle a^p, b^p, c^p \rangle \cong C_{p^{l-1}} \times C_{p^2} \times C_p$  if  $l > 1$ ,  $\Phi(G) = Z(G) \cong C_{p^2} \times C_p$  if  $l = 1$ ,  $G' = \langle b^{p^2}, c^p \rangle$ .
- (B13)  $\langle a, b, c \mid a^{p^{l+1}} = b^{p^m} = c^{p^2} = 1, [b, c] = 1, [c, a] = c^p, [a, b] = a^{p^l} \rangle$ , where  $m \leq 2$ ; moreover,  $|G| = p^{l+m+3}$ ,  $\Phi(G) = Z(G) = \langle a^p, b^p, c^p \rangle \cong C_{p^l} \times C_p^2$  if  $m = 2$ ,  $\Phi(G) = Z(G) \cong C_{p^l} \times C_p$  if  $m = 1$ ,  $G' = \langle a^{p^l}, c^p \rangle$ .
- (B14)  $\langle a, b, c \mid a^{p^{l+1}} = b^{p^{m+1}} = c^{p^n} = 1, [b, c] = 1, [c, a] = b^{p^m}, [a, b] = a^{p^l} \rangle$ , where  $\min\{l, m, n\} = 1$  and  $\max\{m, n\} = 2$ ; moreover,  $|G| = p^{l+m+n+2}$ ,  $\Phi(G) = Z(G) = \langle a^p, b^p, c^p \rangle \cong C_{p^l} \times C_{p^m} \times C_{p^{n-1}}$  if  $n > 1$ ,  $\Phi(G) = Z(G) \cong C_{p^l} \times C_{p^m}$  if  $n = 1$ ,  $G' = \langle a^{p^l}, b^{p^m} \rangle$ .
- (B15)  $\langle a, b, c \mid a^4 = b^4 = c^4 = 1, [b, c] = 1, [c, a] = a^2 c^2, [a, b] = c^2 \rangle$ ; where  $|G| = 2^6$ ,  $\Phi(G) = Z(G) = \langle a^2, b^2, c^2 \rangle \cong C_2^3$ ,  $G' = \langle a^2, c^2 \rangle$ .
- (B16)  $\langle a, b, c \mid a^4 = b^4 = c^4 = 1, [b, c] = 1, [c, a] = a^2 = c^2, [a, b] = b^2 \rangle$ ; where  $|G| = 2^5$ ,  $\Phi(G) = Z(G) = G' = \langle a^2, b^2 \rangle \cong C_2^2$ .
- (B17)  $\langle a, b, c; x \mid a^{p^l} = b^p = c^{p^2} = x^p = 1, [a, b] = x, [a, c] = c^p, [b, c] = [x, a] = [x, b] = [x, c] = 1 \rangle$ , where  $l \geq 2$  if  $p = 2$ ; moreover,  $|G| = p^{l+4}$ ,  $\Phi(G) = Z(G) = \langle a^p, c^p, x \rangle \cong C_{p^{l-1}} \times C_p \times C_p$  if  $l > 1$ ,  $\Phi(G) = Z(G) \cong C_p \times C_p$  if  $l = 1$ ,  $G' = \langle c^p, x \rangle$ .
- (B18)  $\langle a, b, c; x \mid a^{p^l} = b^{p^m} = c^{p^2} = x^p = 1, [a, b] = c^p, [a, c] = x, [b, c] = [x, a] = [x, b] = [x, c] = 1 \rangle$ , where  $l \geq 2$  if  $p = 2$ ,  $m \leq 2$  and  $\min\{l, m\} = 1$ ; moreover,

- $|G| = p^{l+m+3}$ ,  $\Phi(G) = Z(G) = \langle a^p, b^p, c^p, x \rangle \cong C_{p^{l-1}} \times C_p \times C_p$  if  $m = 1$ ,  $\Phi(G) = Z(G) \cong C_{p^{m-1}} \times C_p \times C_p$  if  $l = 1$ ,  $G' = \langle c^p, x \rangle$ .
- (B19)  $\langle a, b, c; x \mid a^{p^{l+1}} = b^{p^m} = c^p = x^p = 1, [a, b] = a^{p^l}, [a, c] = x, [b, c] = [x, a] = [x, b] = [x, c] = 1 \rangle$ , where  $m \leq 2$ ; moreover,  $|G| = p^{l+m+3}$ ,  $\Phi(G) = Z(G) = \langle a^p, b^p, x \rangle \cong C_{p^l} \times C_{p^{m-1}} \times C_p$  if  $m > 1$ ,  $\Phi(G) = Z(G) \cong C_{p^l} \times C_p$  if  $m = 1$ ,  $G' = \langle a^{p^l}, x \rangle$ .
- (B20)  $\langle a, b, c; x \mid a^4 = b^2 = c^4 = x^2 = 1, [a, b] = x, [a, c] = a^2 = c^2, [b, c] = [x, a] = [x, b] = [x, c] = 1 \rangle$ ; where  $|G| = 2^5$ ,  $\Phi(G) = Z(G) = G' = \langle a^2, x \rangle \cong C_2^2$ .

Moreover, Table 2 gives  $(\mu_0, \mu_1, \mu_2)$  and  $\alpha_1(G)$  for the groups (B1)–(B20).

$(\mu_0, \mu_1, \mu_2)$	$\alpha_1(G)$	types of $\mathcal{A}_3$ groups
$(p+1, p^2-1, 1)$	$2p^2-1$	(B1); (B2) where $m = n = 1$ ; (B3) where $n = 1$ ; (B4); (B5) where $n = 1$
$(p+1, p^2-p, p)$	$p^3+p^2-p$	(B2) where $m > 1 = n$ or $n > 1 = m$ ; (B3) where $n > 1$ ; (B5) where $n > 1$
$(1, p^2-1, p+1)$	$p^3+2p^2-1$	(B6); (B9); (B11); (B13) where $p = 2$ and $m = l = 1$ (B17) where $l = 1$ ; (B19) where $p = 2$ and $m = l = 1$
$(1, p^2, p)$	$p^3+p^2$	(B7) where $l > 1$ ; (B10); (B12) where $l > 1$ ; (B13) where $m = 1$ and $l > 1$ (B14) where $n = 1$ or $m = 1$ ; (B16); (B18) where $m = 1$ (B19) where $p = 2$ and $l > 1 = m$ ; (B19) where $p > 2$ and $m = 1$
$(1, p^2+p-1, 1)$	$2p^2+p-1$	(B7) where $l = 1$ ; (B12) where $l = 1$ ; (B13) where $p > 2$ and $m = l = 1$ ; (B20)
$(1, p^2+p-2, 2)$	$3p^2+p-2$	(B8) where $l = 1$
$(1, p^2-p, 2p)$	$2p^3+p^2-p$	(B8) where $l > 1$ ; (B13) where $m = 2$ ; (B14) where $n = m = 2$ (B15); (B17) where $l > 1$ ; (B18) where $m = 2$ ; (B19) where $m = 2$

Table 2: The enumeration of (B1)–(B20)

**Proof** Let  $A$  be an abelian subgroup of index  $p$  and  $B$  be an  $\mathcal{A}_1$ -subgroup of index  $p$ . By Lemma 2.15,  $|G'| \leq p|A'||B'| = p^2$ . It is easy to see that  $\Phi(B) = \Phi(G)$ . By Lemma 2.2,  $\Phi(B) = Z(B)$ . Since  $[\Phi(G), A] = [\Phi(G), B] = 1$  and  $G = AB$ , we have  $\Phi(G) \leq Z(G)$ . Moreover, it is easy to prove that  $G' \leq C_p^2$ . If  $|G'| = p$ , then  $G$  is one of the groups listed in [2, Theorem 3.1]. Since  $G$  has an  $\mathcal{A}_1$ -subgroup of index  $p$ , they are the groups (B1)–(B5). If  $|G'| = p^2$ , then  $G$  is one of the groups listed in [2, Theorem 4.7]. By checking [2, Table 4] we get the groups (B6)–(B20). The correspondence shows as Table 3.

Groups	Groups in [2, Theorem 4.8]	Groups	Groups in [2, Theorem 4.8]
(B6)	(A1) where $l = m = 1$	(B14)	(A10) where $\min\{l, m, n\} = 1$ and $\max\{m, n\} = 2$
(B7)	(A2) where $m = 1$	(B15)	(A11) where $n = 2$
(B8)	(A3) where $m = 1$ and $-\nu \in (F_p^*)^2$	(B16)	(A12) where $h = 1$
(B9)	(A3) where $l = 1, m = 2$ and $-\nu \notin (F_p^*)^2$	(B17)	(B1) where $m = n = 2 = 1$
(B10)	(A4) where $m = 1$	(B18)	(B2) where $n = 1, m \leq 2$ and $\min\{l, m\} = 1$
(B11)	(A6) where $m = 2$	(B19)	(B3) where $n = 1$ and $m \leq 2$
(B12)	(A8) where $m = 2$ and $n = 1$	(B20)	(B4) where $h = 1$
(B13)	(A9) where $n = 1$ and $m \leq 2$		

Table 3: The correspondence from Theorem 4.2 to [2, Theorem 4.7]

We calculate the  $(\mu_0, \mu_1, \mu_2)$  and  $\alpha_1(G)$  of those groups in Theorem 4.2 as follows.

By Lemma 3.18,  $\alpha_1(G) = \mu_1 + p^2\mu_2$ . Hence we only need to calculate  $(\mu_0, \mu_1, \mu_2)$ . Since  $d(G) = 3$ ,  $G$  has  $1 + p + p^2$  maximal subgroups. They are respectively

$$M = \langle b, c, \Phi(G) \rangle;$$

$$M_i = \langle ab^i, c, \Phi(G) \rangle, \text{ where } 0 \leq i \leq p-1;$$



$M_{ij} = \langle ac^i, bc^j, \Phi(G) \rangle$ , where  $0 \leq i, j \leq p-1$ .

**Case 1.**  $G' \cong C_p$ . That is,  $G$  is one of the groups (B1)–(B5).

In this case,  $M$  and  $M_i$  are abelian. Notice that  $|M'_{ij}| = p$ . If  $d(M_{ij}) = 2$ , then, by Lemma 2.2,  $M_{ij} \in \mathcal{A}_1$ . Since  $\Phi(M_{ij}) \leq \Phi(G)$ , we get that  $M_{ij} \in \mathcal{A}_1$  if and only if  $\Phi(G) = \Phi(M_{ij})$ .

By calculations,  $\Phi(G) = \langle a^p, b^p, c^p, G' \rangle$  and  $\Phi(M_{ij}) = \langle a^p c^{ip}, b^p c^{jp}, G' \rangle$ . Since  $\Phi(G)/G' = \langle \bar{a}^p \rangle \times \langle \bar{b}^p \rangle \times \langle \bar{c}^p \rangle$  where  $\bar{a}^p = 1$  or  $\bar{b}^p = 1$ , and  $\langle \bar{c}^p \rangle \cong C_p$ , the following conclusions hold:

- (i) If  $\bar{a}^p = 1$  and  $\bar{b}^p \neq 1$ , then  $\Phi(G) = \Phi(M_{ij})$  if and only if  $(i, p) = 1$ ;
- (ii) If  $\bar{a}^p \neq 1$  and  $\bar{b}^p = 1$ , then  $\Phi(G) = \Phi(M_{ij})$  if and only if  $(j, p) = 1$ ;
- (iii) If  $\bar{a}^p = \bar{b}^p = 1$ , then  $\Phi(G) = \Phi(M_{ij})$  if and only if  $(i, p) = 1$  or  $(j, p) = 1$ .

If  $|G| = p^5$ , then case (iii) happens. In this case,  $M_{00}$  is the unique  $\mathcal{A}_2$ -subgroup of  $G$ . Hence  $(\mu_0, \mu_1, \mu_2) = (p+1, p^2-1, 1)$ .

If  $|G| > p^5$ , then case (i) or (ii) happens. If (i) happens, then  $M_{ij} \in \mathcal{A}_2$  if and only if  $i = 0$ . If (ii) happens, then  $M_{ij} \in \mathcal{A}_2$  if and only if  $j = 0$ . Hence  $(\mu_0, \mu_1, \mu_2) = (p+1, p^2-p, p)$ .

**Case 2.**  $G' \cong C_p^2$ . That is,  $G$  is one of the groups (B6)–(B20).

In this case,  $M$  is the unique abelian subgroup of index  $p$ . Notice that  $|M'_i| = p$ . If  $d(M_i) = 2$ , then, by Lemma 2.2,  $M_i \in \mathcal{A}_1$ . Since  $\Phi(M_i) \leq \Phi(G)$ , we get that  $M_i \in \mathcal{A}_1$  if and only if  $\Phi(G) = \Phi(M_{ij})$ . Similar argument gives that  $M_{ij} \in \mathcal{A}_1$  if and only if  $\Phi(G) = \Phi(M_{ij})$ .

Subcase 1.  $G$  is the group (B6).

In this case,  $M_0 = \langle a, c \rangle \times \langle b^p \rangle \in \mathcal{A}_2$ . For  $i \neq 0$ ,  $M_i = \langle ab^i, c \rangle \in \mathcal{A}_1$ . By calculations,  $\Phi(G) = \langle b^p, c^p \rangle$  and  $\Phi(M_{ij}) = \langle c^{ip}, b^p c^{jp} \rangle$ . Hence  $\Phi(G) = \Phi(M_{ij})$  if and only if  $(i, p) = 1$ . In Summary,  $G$  has  $p+1$   $\mathcal{A}_2$ -subgroups. They are  $M_0$  and  $M_{0j}$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2-1, p+1)$ .

Subcase 2.  $G$  is the group (B7).

In this case,  $M_i = \langle ab^i, c \rangle \in \mathcal{A}_1$ . By calculations,  $\Phi(G) = \langle a^p, b^p, c^p \rangle$  and  $\Phi(M_{ij}) = \langle a^p c^{ip}, b^p c^{jp}, b^{-p} b^{-jp} c^{-jp} \rangle = \langle a^p c^{ip}, b^p c^{jp}, b^{-jp} \rangle$ .

If  $l = 1$ , then  $a^p = 1$ . In this case,  $\Phi(G) = \Phi(M_{ij})$  if and only if  $(i, p) = 1$  or  $(j, p) = 1$ . That is,  $G$  has a unique  $\mathcal{A}_2$ -subgroup  $M_{00}$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2+p-1, 1)$ .

If  $l > 1$ , then  $a^p \neq 1$ . In this case,  $\Phi(G) = \Phi(M_{ij})$  if and only if  $(j, p) = 1$ . That is,  $G$  has  $p$   $\mathcal{A}_2$ -subgroups  $M_{i0}$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2, p)$ .

Subcase 3.  $G$  is the group (B8).

In this case,  $M_i = \langle ab^i, c \rangle \in \mathcal{A}_1$ . By calculations,  $\Phi(G) = \langle a^p, b^p, c^p \rangle$  and  $\Phi(M_{ij}) = \langle a^p c^{ip}, b^p c^{jp}, b^{-tp} c^{\nu p} b^{-jp} c^{-tjp} \rangle = \langle a^p c^{ip}, b^p c^{jp}, c^{\nu p} b^{-jp} \rangle$ .

If  $l = 1$ , then  $a^p = 1$ . In this case,  $\Phi(G) = \Phi(M_{ij})$  if and only if  $(i, p) = 1$  or  $\begin{vmatrix} 1 & j \\ -j & \nu \end{vmatrix} \neq 0$ . That is,  $G$  has 2  $\mathcal{A}_2$ -subgroups  $M_{0j}$  where  $j^2 = -\nu$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2 + p - 2, 2)$ .

If  $l > 1$ , then  $a^p \neq 1$ . In this case,  $\Phi(G) = \Phi(M_{ij})$  if and only if  $\begin{vmatrix} 1 & j \\ -j & \nu \end{vmatrix} \neq 0$ . That is,  $G$  has  $2p$   $\mathcal{A}_2$ -subgroups  $M_{ij}$  where  $j^2 = -\nu$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2 - p, 2p)$ .

Subcase 4.  $G$  is the group (B9).

In this case,  $M_0 = \langle a, c \rangle * \langle b^p c^{tp} \rangle \in \mathcal{A}_2$ . For  $i \neq 0$ ,  $M_i = \langle ab^i, c \rangle \in \mathcal{A}_1$ . By calculations,  $\Phi(G) = \langle b^p, c^p \rangle$  and  $\Phi(M_{ij}) = \langle c^{ip}, b^p c^{jp}, G' \rangle$ . Hence  $\Phi(G) = \Phi(M_{ij})$  if and only if  $(i, p) = 1$ . In summary,  $G$  has  $p$   $\mathcal{A}_2$ -subgroups. They are  $M_0$  and  $M_{0j}$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2 - 1, p + 1)$ .

Subcase 5.  $G$  is the group (B10).

In this case,  $M_i = \langle ab^i, c \rangle \in \mathcal{A}_1$ . By calculations,  $\Phi(G) = \langle a^2, b^2, c^2 \rangle$  and  $\Phi(M_{ij}) = \langle a^2 c^{2i} b^{2i}, b^2 c^{2j}, c^2 b^{2j} \rangle$ . Hence  $\Phi(G) = \Phi(M_{ij})$  if and only if  $j = 0$ . In summary,  $G$  has  $p$   $\mathcal{A}_2$ -subgroups. They are  $M_{i1}$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2, p)$ .

Subcase 6.  $G$  is the group (B11).

In this case,  $M_0 = \langle a, c \rangle * \langle b^2 \rangle \in \mathcal{A}_2$  and  $M_1 = \langle ab, c \rangle \in \mathcal{A}_1$ . By calculations,  $\Phi(G) = \langle b^2, c^2 \rangle$  and  $\Phi(M_{ij}) = \langle c^{2i}, b^2 c^{2j}, G' \rangle$ . Hence  $\Phi(G) = \Phi(M_{ij})$  if and only if  $i = 1$ . In summary,  $G$  has  $p + 1$   $\mathcal{A}_2$ -subgroups. They are  $M_0$  and  $M_{0j}$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2 - 1, p + 1)$ .

Subcase 7.  $G$  is the group (B12).

By calculations,  $\Phi(M_{ij}) = \langle a^p c^{ip}, b^p c^{jp}, c^{\nu p} b^{-jp^2} \rangle = \langle a^p, b^p, c^p \rangle = \Phi(G)$ . Hence  $M_{ij} \in \mathcal{A}_1$ . Notice that  $\Phi(M_i) = \langle a^p b^{ip}, c^p, b^{p^2} \rangle$ .

If  $l = 1$ , then  $a^p = 1$ . In this case,  $\Phi(G) = \Phi(M_i)$  if and only if  $(i, p) = 1$ . That is,  $G$  has a unique  $\mathcal{A}_2$ -subgroup  $M_0$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2 + p - 1, 1)$ .

If  $l > 1$ , then  $a^p \neq 1$  and  $\Phi(G) > \Phi(M_i)$ . Hence  $M_i \in \mathcal{A}_2$  and  $(\mu_0, \mu_1, \mu_2) = (1, p^2, p)$ .

Subcase 8.  $G$  is the group (B13).

By calculations,  $\Phi(G) = \langle a^p, b^p, c^p \rangle$ . If  $p > 2$ , then  $\Phi(M_i) = \langle a^p b^{ip}, c^p \rangle$ . Hence  $M_i \in \mathcal{A}_2$  for  $m = 2$  and  $M_i \in \mathcal{A}_1$  for  $m = 1$ . If  $p = 2$ , then  $\Phi(M_i) = \langle a^2 b^{2i} a^{i2^l}, c^2 \rangle$ . Hence  $M_i \in \mathcal{A}_2$  for  $m = 2$  and  $M_i \in \mathcal{A}_1$  for  $m = 1 \leq l$ . If  $l = m = 1$  and  $p = 2$ , then  $M_0 \in \mathcal{A}_1$  and  $M_1 \in \mathcal{A}_2$ .

If  $p > 2$ , then  $\Phi(M_{ij}) = \langle a^p c^{ip}, b^p c^{jp}, a^{p^l} c^{-jp} \rangle$ . We discuss the value of  $m$  and  $l$ .

If  $m = 2$  and  $l > 1$ , then  $\Phi(M_{ij}) = \langle a^p c^{ip}, b^p c^{jp}, c^{-jp} \rangle$ . In this case,  $\Phi(G) = \Phi(M_{ij})$  if and only if  $(j, p) = 1$  and  $G$  has  $2p$   $\mathcal{A}_2$ -subgroups  $M_i$  and  $M_{i0}$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2 - p, 2p)$ .

If  $m = 2$  and  $l = 1$ , then  $\Phi(M_{ij}) = \langle a^p c^{ip}, b^p c^{jp}, c^{(i+j)p} \rangle$ . In this case,  $\Phi(G) = \Phi(M_{ij})$  if and only if  $(i + j, p) = 1$  and  $G$  has  $2p$   $\mathcal{A}_2$ -subgroups  $M_i$  and  $M_{ij}$  where

$i + j = p$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2 - p, 2p)$ .

If  $m = 1$  and  $l > 1$ , then  $\Phi(M_{ij}) = \langle a^p c^{ip}, c^{jp} \rangle$ . In this case,  $\Phi(G) = \Phi(M_{ij})$  if and only if  $(j, p) = 1$  and  $G$  has  $p$   $\mathcal{A}_2$ -subgroups  $M_{i0}$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2, p)$ .

If  $l = m = 1$ , then  $\Phi(M_{ij}) = \langle a^p c^{ip}, c^{jp}, a^p c^{-jp} \rangle = \langle a^p, c^{ip}, c^{jp} \rangle$ . In this case,  $\Phi(G) = \Phi(M_{ij})$  if and only if  $(i, p) = 1$  or  $(j, p) = 1$ , and  $G$  has a unique  $\mathcal{A}_2$ -subgroup  $M_{00}$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2 + p - 1, 1)$ .

If  $p = 2$ , then  $\Phi(M_{ij}) = \langle a^2, b^2 c^{2j}, a^{2l} c^{-2j} \rangle = \langle a^2, b^2, c^{-2j} \rangle$ . In this case,  $\Phi(G) = \Phi(M_{ij})$  if and only if  $(j, 2) = 1$  and  $M_{i0} \in \mathcal{A}_2$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2 - p, 2p)$  for  $m = 2$ ,  $(\mu_0, \mu_1, \mu_2) = (1, p^2, p)$  for  $m = 1 < l$ , and  $(\mu_0, \mu_1, \mu_2) = (1, p^2 - 1, p + 1)$  for  $l = m = 1$ .

Subcase 9.  $G$  is the group (B14).

By calculations,  $\Phi(G) = \langle a^p, b^p, c^p \rangle$ . If  $p > 2$ , then  $\Phi(M_i) = \langle a^p b^{ip}, c^p, b^{p^m} \rangle$ . Hence  $M_i \in \mathcal{A}_2$  for  $m = 2$  and  $M_i \in \mathcal{A}_1$  for  $m = 1$ . If  $p = 2$ , then  $\Phi(M_i) = \langle a^2 b^{2i} a^{i2^l}, c^2, b^{2^m} \rangle$ . Hence  $M_i \in \mathcal{A}_2$  for  $m = 2$  and  $M_i \in \mathcal{A}_1$  for  $m = 1 \leq l$ . If  $l = m = 1$  and  $p = 2$ , then  $M_0 \in \mathcal{A}_1$  and  $M_1 \in \mathcal{A}_2$ .

If  $p > 2$ , then  $\Phi(M_{ij}) = \langle a^p c^{ip}, b^p c^{jp}, a^{p^l} b^{-jp^m} \rangle$ . We discuss the value of  $m$  and  $l$ .

If  $n = 1$ , then  $\Phi(M_{ij}) = \langle a^p, b^p \rangle = \Phi(G)$  and hence  $\Phi(M_{ij}) \in \mathcal{A}_1$ . Since  $\max\{m, n\} = 2$ ,  $m = 2$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2, p)$ .

If  $m = n = 2$ , then  $l = 1$  since  $\min\{l, m, n\} = 1$ . Notice that  $\Phi(M_{ij}) = \langle c^{ip}, b^p c^{jp}, a^p \rangle$ . In this case,  $\Phi(G) = \Phi(M_{ij})$  if and only if  $(i, p) = 1$  and  $G$  has  $2p$   $\mathcal{A}_2$ -subgroups  $M_i$  and  $M_{0j}$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2 - p, 2p)$ .

If  $n = 2$  and  $l = m = 1$ , then  $\Phi(M_{ij}) = \langle a^p c^{ip}, b^p c^{jp}, a^p b^{-jp} \rangle$ . In this case,  $\Phi(G) = \Phi(M_{ij})$  if and only if  $j^2 - i \neq 0$  and  $G$  has  $p$   $\mathcal{A}_2$ -subgroups  $M_{ij}$  where  $i = j^2$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2, p)$ .

If  $n = 2$  and  $m = 1 < l$ , then  $\Phi(M_{ij}) = \langle a^p c^{ip}, b^p c^{jp}, b^{-jp} \rangle$ . In this case,  $\Phi(G) = \Phi(M_{ij})$  if and only if  $(j, p) = 1$  and  $G$  has  $p$   $\mathcal{A}_2$ -subgroups  $M_{i0}$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2, p)$ .

If  $p = 2$ , then  $\Phi(M_{ij}) = \langle a^2 c^{2i} b^{i2^m}, b^2 c^{2j}, a^{2l} b^{-j2^m} \rangle$ . We discuss the value of  $m$  and  $l$ .

If  $n = 1$ , then  $\Phi(M_{ij}) = \langle a^2, b^2 \rangle = \Phi(G)$  and hence  $\Phi(M_{ij}) \in \mathcal{A}_1$ . Since  $\max\{m, n\} = 2$ ,  $m = 2$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2, p)$ .

If  $m = n = 2$ , then  $l = 1$  since  $\min\{l, m, n\} = 1$ . Notice that  $\Phi(M_{ij}) = \langle c^{2i}, b^2 c^{2j}, a^2 \rangle$ . In this case,  $\Phi(G) = \Phi(M_{ij})$  if and only if  $(i, 2) = 1$  and  $G$  has  $2p$   $\mathcal{A}_2$ -subgroups  $M_i$  and  $M_{0j}$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2 - p, 2p)$ .

If  $n = 2$  and  $l = m = 1$ , then  $\Phi(M_{ij}) = \langle a^2 c^{2i} b^{2i}, b^2 c^{2j}, a^2 b^{2j} \rangle$ . In this case,  $\Phi(G) = \Phi(M_{ij})$  if and only if  $i = 1$  or  $j = 1$ , and  $G$  has 2  $\mathcal{A}_2$ -subgroups  $M_1$  and  $M_{00}$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2, p)$ .

If  $n = 2$  and  $m = 1 < l$ , then  $\Phi(M_{ij}) = \langle a^2 c^{2i} b^{2i}, b^2 c^{2j}, b^{-2j} \rangle$ . In this case,  $\Phi(G) = \Phi(M_{ij})$  if and only if  $(j, 2) = 1$  and  $G$  has 2  $\mathcal{A}_2$ -subgroups  $M_{i0}$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2, p)$ .

Subcase 10.  $G$  is the group (B15).

By calculations,  $\Phi(G) = \langle a^2, b^2, c^2 \rangle$ ,  $M_0 = \langle c, a \rangle \times \langle b^2 \rangle \in \mathcal{A}_2$  and  $M_1 = \langle c, ab \rangle \in \mathcal{A}_1$ . Since  $\Phi(M_{ij}) = \langle a^{2(1+i)}, b^2 c^{2j}, c^{2(1+j)} a^{2j} \rangle$ ,  $\Phi(G) = \Phi(M_{ij})$  if and only if  $i = j = 0$ . Hence  $G$  has 4  $\mathcal{A}_2$ -subgroups  $M_0$  and  $M_{ij}$  where  $i \neq 0$  or  $j \neq 0$ , and  $(\mu_0, \mu_1, \mu_2) = (1, 2, 4)$ .

Subcase 11.  $G$  is the group (B16).

By calculations,  $\Phi(G) = \langle a^2, b^2 \rangle$ ,  $M_i = \langle c, ab^i \rangle \times \langle b^2 \rangle \in \mathcal{A}_2$ . Since  $\Phi(M_{ij}) = \langle a^2 = c^2, b^2 c^{2j}, b^2 a^{2j} \rangle = \Phi(G)$ ,  $M_{ij} \in \mathcal{A}_1$ . Hence  $G$  has  $p$   $\mathcal{A}_2$ -subgroups  $M_i$  and  $(\mu_0, \mu_1, \mu_2) = (1, p^2, p)$ .

Subcase 12.  $G$  is the group (B17).

By calculations,  $\Phi(G) = \langle a^p, c^p, x \rangle$  and  $M_i = \langle ab^i, c \rangle \times \langle x \rangle \in \mathcal{A}_2$ .

If  $p > 2$ , then  $\Phi(M_{ij}) = \langle a^p c^{ip}, c^{jp}, xc^{-jp} \rangle = \langle a^p c^{ip}, c^{jp}, x \rangle$ . In this case,  $\Phi(G) = \Phi(M_{ij})$  if and only if  $(j, p) = 1$  and  $G$  has  $2p$   $\mathcal{A}_2$ -subgroups  $M_i$  and  $M_{i0}$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2 - p, 2p)$ .

If  $p = 2$ , then  $\Phi(M_{ij}) = \langle a^2, c^{2j}, xc^{2j} \rangle = \langle a^2, c^{2j}, x \rangle$ . In this case,  $\Phi(G) = \Phi(M_{ij})$  if and only if  $(j, 2) = 1$  and  $G$  has 4  $\mathcal{A}_2$ -subgroups  $M_i$  and  $M_{i0}$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2 - p, 2p)$ .

Subcase 13.  $G$  is the group (B18).

If  $l = 1$  and  $m = 2$ , then  $p > 2$  and  $\Phi(G) = \langle b^p, c^p, x \rangle$ . In addition,  $\Phi(M_i) = \langle b^{ip}, c^p, x \rangle$  and  $\Phi(M_{ij}) = \langle c^{ip}, b^p c^{jp}, c^p x^{-j} \rangle$ . In this case,  $\Phi(M_i) = \Phi(G)$  if and only if  $(i, p) = 1$ , and  $\Phi(M_{ij}) = \Phi(G)$  if and only if  $(i, p) = 1$  and  $(j, p) = 1$ . Hence  $G$  has  $2p$   $\mathcal{A}_2$ -subgroups  $M_0$ ,  $M_{0j}$  and  $M_{i0}$ , and  $(\mu_0, \mu_1, \mu_2) = (1, p^2 - p, 2p)$ .

If  $l = m = 1$ , then  $p > 2$  and  $\Phi(G) = \langle c^p, x \rangle$ . Since  $\Phi(M_i) = \langle c^p, x \rangle = \Phi(G)$ ,  $M_i \in \mathcal{A}_1$ . Since  $\Phi(M_{ij}) = \langle c^{ip}, c^{jp}, c^p x^{-j} \rangle$ ,  $\Phi(M_{ij}) = \Phi(G)$  if and only if  $(j, p) = 1$ . Hence  $G$  has  $p$   $\mathcal{A}_2$ -subgroups  $M_{i0}$ , and  $(\mu_0, \mu_1, \mu_2) = (1, p^2, p)$ .

If  $l > 1$ , then  $m = 1$  since  $\min\{l, m\} = 1$ . In this case,  $\Phi(G) = \langle a^p, c^p, x \rangle$ . In addition, if  $p > 2$ , then  $\Phi(M_i) = \langle a^p, c^p, x \rangle$  and  $\Phi(M_{ij}) = \langle a^p c^{ip}, c^{jp}, c^p x^{-j} \rangle$ . In this case,  $M_i \in \mathcal{A}_1$  and  $\Phi(M_{ij}) = \Phi(G)$  if and only if  $(j, p) = 1$ . Hence  $G$  has  $p$   $\mathcal{A}_2$ -subgroups  $M_{i0}$  and  $(\mu_0, \mu_1, \mu_2) = (1, p^2, p)$ ; if  $p = 2$ , then  $\Phi(M_i) = \langle a^2 c^{2i}, c^2, x \rangle$  and  $\Phi(M_{ij}) = \langle a^2 c^{2i} x^i, c^{2j}, c^2 x^j \rangle$ . In this case,  $M_i \in \mathcal{A}_1$  and  $\Phi(M_{ij}) = \Phi(G)$  if and only if  $(j, 2) = 1$ . Hence  $G$  has 2  $\mathcal{A}_2$ -subgroups  $M_{i0}$  and  $(\mu_0, \mu_1, \mu_2) = (1, p^2, p)$ .

Subcase 14.  $G$  is the group (B19).

If  $m = 2$ , then  $\Phi(G) = \langle a^p, b^p, x \rangle$ . In addition, if  $p > 2$ , then  $\Phi(M_i) = \langle a^p b^{ip}, x \rangle$  and  $\Phi(M_{ij}) = \langle a^p, b^p, a^{p^l} x^{-j} \rangle$ . In this case,  $\Phi(M_i) \in \mathcal{A}_2$  and  $\Phi(M_{ij}) = \Phi(G)$  if and only if  $(j, p) = 1$ . Hence  $G$  has  $2p$   $\mathcal{A}_2$ -subgroups  $M_i$  and  $M_{i0}$ , and  $(\mu_0, \mu_1, \mu_2) = (1, p^2 - p, 2p)$ ; if  $p = 2$ , then  $\Phi(M_i) = \langle a^2 b^{2i} a^{i2^l}, x \rangle$  and  $\Phi(M_{ij}) = \langle a^2 x^i, b^2, a^{2^l} x^{-j} \rangle$ . In this case,  $\Phi(M_i) \in \mathcal{A}_2$ .  $\Phi(M_{ij}) = \Phi(G)$  if and only if  $(j, 2) = 1$  for  $l > 1$ . Hence  $G$  has  $2p$   $\mathcal{A}_2$ -subgroups  $M_i$  and  $M_{i0}$  for  $l > 1$ .  $\Phi(M_{ij}) = \Phi(G)$  if and only if  $i \neq j$  for  $l = 1$ .

Hence  $G$  has  $2p$   $\mathcal{A}_2$ -subgroups  $M_i$  and  $M_{ij}$  where  $i \neq j$  for  $l > 1$ . In summary,  $(\mu_0, \mu_1, \mu_2) = (1, p^2 - p, 2p)$ .

If  $l = m = 1$ , then  $\Phi(G) = \langle a^p, x \rangle$ . In addition, if  $p > 2$ , then  $\Phi(M_i) = \langle a^p, x \rangle$  and  $\Phi(M_{ij}) = \langle a^p, a^p x^{-j} \rangle$ . In this case,  $\Phi(M_i) \in \mathcal{A}_1$  and  $\Phi(M_{ij}) = \Phi(G)$  if and only  $(j, p) = 1$ . Hence  $G$  has  $p$   $\mathcal{A}_2$ -subgroups  $M_{i0}$ , and  $(\mu_0, \mu_1, \mu_2) = (1, p^2, p)$ ; if  $p = 2$ , then  $\Phi(M_i) = \langle a^2 a^{i^2}, x \rangle$  and  $\Phi(M_{ij}) = \langle a^2 x^i, a^2 x^{-j} \rangle$ . In this case,  $\Phi(M_0) \in \mathcal{A}_1$ ,  $\Phi(M_1) \in \mathcal{A}_2$  and  $\Phi(M_{ij}) = \Phi(G)$  if and only  $i \neq j$ . Hence  $G$  has  $p + 1$   $\mathcal{A}_2$ -subgroups  $M_1$  and  $M_{ij}$  where  $i \neq j$ , and  $(\mu_0, \mu_1, \mu_2) = (1, p^2 - 1, p + 1)$ .

If  $l > 1 = m$ , then  $\Phi(G) = \langle a^p, x \rangle$ . In addition, if  $p > 2$ , then  $\Phi(M_i) = \langle a^p, x \rangle$  and  $\Phi(M_{ij}) = \langle a^p, a^{p^l} x^{-j} \rangle$ . In this case,  $\Phi(M_i) \in \mathcal{A}_1$  and  $\Phi(M_{ij}) = \Phi(G)$  if and only  $(j, p) = 1$ . Hence  $G$  has  $p$   $\mathcal{A}_2$ -subgroups  $M_{i0}$ , and  $(\mu_0, \mu_1, \mu_2) = (1, p^2, p)$ ; if  $p = 2$ , then  $\Phi(M_i) = \langle a^2 a^{i^{2^l}}, x \rangle$  and  $\Phi(M_{ij}) = \langle a^2 x^i, a^{2^l} x^{-j} \rangle$ . In this case,  $\Phi(M_i) \in \mathcal{A}_1$  and  $\Phi(M_{ij}) = \Phi(G)$  if and only  $(j, 2) = 1$ . Hence  $G$  has  $p$   $\mathcal{A}_2$ -subgroups  $M_{i0}$ , and  $(\mu_0, \mu_1, \mu_2) = (1, p^2, p)$ .

Subcase 15.  $G$  is the group (B20).

By calculations,  $\Phi(G) = \langle a^2 = c^2, x \rangle$ ,  $M_0 = \langle a, c \rangle \times \langle x \rangle \in \mathcal{A}_2$ , and  $M_1 = \langle ab, c \rangle \in \mathcal{A}_1$ . Since  $\Phi(M_{ij}) = \langle a^2, c^{2j}, xc^{2j} \rangle = \langle a^2, x \rangle = \Phi(G)$ ,  $M_{ij} \in \mathcal{A}_1$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, p^2 + p - 1, 1)$ .

To sum up, we get the Table 3. □

**Lemma 4.3.** *Suppose that  $G$  is a finite  $p$ -group such that  $\Phi(G') = 1$ ,  $C_p^2 \cong G_3 \leq Z(G)$  and  $G/G_3 \cong M_p(1, 1, 1)$ , where  $p \geq 3$ . Let  $G = \langle a, b, c, x, y \rangle$  such that*

$$a^p = x^{w_{11}} y^{w_{12}}, b^p = x^{w_{21}} y^{w_{22}}, c^p = x^p = y^p = 1, [a, b] = c, [c, a] = x, [c, b] = y,$$

$N = \langle c, b, x, y \rangle$  and  $M_i = \langle ab^i, c, x, y \rangle$ . Then  $N$  and  $M_i$  are all maximal subgroups of  $G$ , and the following conclusions hold:

- (1)  $N \in \mathcal{A}_2$  if and only if  $w_{21} = 0$ ;
- (2) If  $p > 3$ , then  $M_i \in \mathcal{A}_2$  if and only if  $w_{12} + i(w_{22} - w_{11}) - i^2 w_{21} = 0$ ;
- (3) If  $p = 3$ , then  $M_i \in \mathcal{A}_2$  if and only if  $w_{12} + i(w_{22} - w_{11}) - i^2 w_{21} - i^2 = 0$ ;
- (4)  $\alpha_1(G) = \mu_1 + \mu_2 p^2$ .

**Proof** If  $w_{21} = 0$ , then  $N = \langle c, b \rangle \times \langle x \rangle \in \mathcal{A}_2$ . If  $w_{21} \neq 0$ , then  $N = \langle c, b \rangle \in \mathcal{A}_1$ . Hence (1) holds.

It is easy to see that  $M_i \in \mathcal{A}_2$  if and only if  $\langle (ab^i)^p \rangle = \langle [c, ab^i] \rangle = \langle xy^i \rangle$ .

If  $p > 3$ , then  $(ab^i)^p = a^p b^{ip} = x^{w_{11} + iw_{21}} y^{w_{12} + iw_{22}}$ . Hence  $M_i \in \mathcal{A}_2$  if and only if  $w_{12} + iw_{22} = i(w_{11} + iw_{21})$ . The later is  $w_{12} + i(w_{22} - w_{11}) - i^2 w_{21} = 0$ . Hence (2) holds.

If  $p = 3$ , then  $(ab^i)^3 = a^3 b^{3i} [a, b^{-i}, a] [a, b^{-i}, b^{-i}] = x^{w_{11} + iw_{21} - i} y^{w_{12} + iw_{22} + i^2}$ . Hence  $M_i \in \mathcal{A}_2$  if and only if  $w_{12} + iw_{22} + i^2 = i(w_{11} + iw_{21} - i)$ . The later is  $w_{12} + i(w_{22} - w_{11}) - i^2 w_{21} - i^2 = 0$ . Hence (3) holds.

Let  $M \in \Gamma_1$ . Then, by above argument,  $|M'| = p$ . If  $M \in \mathcal{A}_2$ , then, by Lemma 2.6 (7),  $\alpha_1(M) = p^2$ . By Hall's enumeration principle,  $\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = \mu_1 + p^2\mu_2$ . Hence (4) holds.  $\square$

**Theorem 4.4.**  *$G$  has no abelian subgroup of index  $p$ ,  $G$  has at least two distinct  $\mathcal{A}_1$ -subgroups of index  $p$  and  $d(G) = 2$  if and only if  $G$  is isomorphic to one of the following pairwise non-isomorphic groups:*

(Ci)  $c(G) = 3$  and  $G' \cong C_2^2$ . In this case,  $(\mu_0, \mu_1, \mu_2) = (0, 2, 1)$  and  $\alpha_1(G) = 6$ .

(C1)  $\langle a, b; c, d \mid a^4 = b^2 = c^2 = d^2 = 1, [a, b] = c, [c, a] = d, [c, b] = [d, a] = [d, b] = 1 \rangle$ ; where  $|G| = 2^5$ ,  $\Phi(G) = \langle a^2, c, d \rangle \cong C_2^3$ ,  $G' = \langle c, d \rangle$  and  $Z(G) = \langle d \rangle \cong C_2$ .

(C2)  $\langle a, b; c \mid a^8 = b^2 = c^2 = 1, [a, b] = c, [c, a] = a^4, [c, b] = 1 \rangle$ ; where  $|G| = 2^5$ ,  $\Phi(G) = \langle a^2, c \rangle \cong C_4 \times C_2$ ,  $G' = \langle c, a^4 \rangle$  and  $Z(G) = \langle a^4 \rangle \cong C_2$ .

(C3)  $\langle a, b; c \mid a^8 = c^2 = 1, b^2 = a^4, [a, b] = c, [c, a] = [a^2, b] = b^2, [c, b] = 1 \rangle$ ; where  $|G| = 2^5$ ,  $\Phi(G) = \langle a^2, c \rangle \cong C_4 \times C_2$ ,  $G' = \langle c, a^4 \rangle$  and  $Z(G) = \langle a^4 \rangle \cong C_2$ .

(C4)  $\langle a, b; c \mid a^{2^{n+1}} = b^2 = c^2 = 1, [a, b] = c, [c, a] = a^{2^n}, [c, b] = 1 \rangle$ , where  $n \geq 3$ ; moreover,  $|G| = 2^{n+3}$ ,  $\Phi(G) = \langle a^2, c \rangle \cong C_{2^n} \times C_2$ ,  $G' = \langle c, a^{2^n} \rangle$  and  $Z(G) = \langle a^4 \rangle \cong C_{2^{n-1}}$ .

(C5)  $\langle a, b; c \mid a^{2^n} = b^4 = c^2 = 1, [a, b] = c, [c, a] = b^2, [c, b] = 1 \rangle$ , where  $n \geq 3$ ; moreover,  $|G| = 2^{n+3}$ ,  $\Phi(G) = \langle a^2, b^2, c \rangle \cong C_{2^{n-1}} \times C_2 \times C_2$ ,  $G' = \langle c, b^2 \rangle$  and  $Z(G) = \langle a^4, b^2 \rangle \cong C_{2^{n-2}} \times C_2$ .

(C6)  $\langle a, b; c, d \mid a^{2^n} = b^2 = c^2 = d^2 = 1, [a, b] = c, [c, a] = d, [c, b] = 1, [d, a] = [d, b] = 1 \rangle$ , where  $n \geq 3$ ; moreover,  $|G| = 2^{n+3}$ ,  $\Phi(G) = \langle a^2, c, d \rangle \cong C_{2^{n-1}} \times C_2 \times C_2$ ,  $G' = \langle c, d \rangle$  and  $Z(G) = \langle a^4, d \rangle \cong C_{2^{n-2}} \times C_2$ .

(Cii)  $\Phi(G') \leq G_3 \cong C_p^2$ .

(C7)  $\langle a, b; c, d, e \mid a^3 = b^3 = c^3 = d^3 = e^3 = 1, [a, b] = c, [c, a] = d, [c, b] = e, [d, a] = [d, b] = [e, a] = [e, b] = 1 \rangle$ , where  $|G| = 3^5$ ,  $\Phi(G) = G' = \langle c, d, e \rangle$  and  $Z(G) = \langle d, e \rangle \cong C_3^2$ .

(C8)  $\langle a, b; c, d \mid a^9 = c^3 = d^3 = 1, b^3 = a^3, [a, b] = c, [c, a] = d, [c, b] = a^3, [d, a] = [d, b] = 1 \rangle$ ; where  $|G| = 3^5$ ,  $\Phi(G) = G' = \langle a^3, c, d \rangle$ ,  $Z(G) = \langle a^3, d \rangle \cong C_3^2$ .

(C9)  $\langle a, b; c, d \mid a^9 = b^3 = c^3 = d^3 = 1, [a, b] = c, [c, a] = d, [c, b] = a^{-3}, [d, a] = [d, b] = 1 \rangle$ ; where  $|G| = 3^5$ ,  $\Phi(G) = G' = \langle a^3, c, d \rangle$  and  $Z(G) = \langle a^3, d \rangle \cong C_3^2$ .

(C10)  $\langle a, b; c \mid a^9 = b^9 = c^3 = 1, [a, b] = c, [c, a] = a^3, [c, b] = b^3 \rangle$ ; where  $|G| = 3^5$ ,  $\Phi(G) = G' = \langle a^3, b^3, c \rangle$  and  $Z(G) = \langle a^3, b^3 \rangle \cong C_3^2$ .

(C11)  $\langle a, b, c \mid a^{p^2} = b^{p^2} = c^p = 1, [a, b] = c, [c, a] = a^p b^{\nu p}, [c, b] = b^p \rangle$ , where  $p > 3$ ,  $\nu = 1$  or a fixed quadratic non-residue modular  $p$ ; moreover,  $|G| = p^5$ ,  $\Phi(G) = G' = \langle a^p, b^p, c \rangle$  and  $Z(G) = \langle a^p, b^p \rangle \cong C_p^2$ .

- (C12)  $\langle a, b, c \mid a^{p^2} = b^{p^2} = c^p = 1, [a, b] = c, [c, a] = b^{\nu p}, [c, b] = a^{-p}, [a^p, b] = 1 \rangle$ , where  $p > 3$ ,  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$  such that  $-\nu \in F_p^2$ ; moreover,  $|G| = p^5$ ,  $\Phi(G) = G' = \langle a^p, b^p, c \rangle$  and  $Z(G) = \langle a^p, b^p \rangle \cong C_p^2$ .
- (C13)  $\langle a, b, c \mid a^{p^2} = b^{p^2} = c^p = 1, [a, b] = c, [c, a]^{1+r} = a^p b^p, [c, b]^{1+r} = a^{-r p} b^p, [a^p, b] = 1 \rangle$ , where  $p > 3$ ,  $r \neq 0, -1$  and  $-r \in (F_p^*)^2$ ; moreover,  $|G| = p^5$ ,  $\Phi(G) = G' = \langle a^p, b^p, c \rangle$  and  $Z(G) = \langle a^p, b^p \rangle \cong C_p^2$ .
- (C14)  $\langle a, b, c, d \mid a^{p^2} = b^p = c^p = d^p = 1, [a, b] = c, [c, a] = a^p, [c, b] = d \rangle$ , where  $p > 3$ ; moreover,  $|G| = p^5$ ,  $\Phi(G) = G' = \langle a^p, c, d \rangle$  and  $Z(G) = \langle a^p, d \rangle \cong C_p^2$ .
- (C15)  $\langle a, b, c, d \mid a^p = b^{p^2} = c^p = d^p = 1, [a, b] = c, [c, a] = b^{\nu p}, [c, b] = d, [d, a] = [d, b] = 1 \rangle$ , where  $p > 3$ ,  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ ; moreover,  $|G| = p^5$ ,  $\Phi(G) = G' = \langle b^p, c, d \rangle$  and  $Z(G) = \langle b^p, d \rangle \cong C_p^2$ .
- (C16)  $\langle a, b, c \mid a^8 = c^4 = 1, b^2 = a^4, [a, b] = c, [c, a] = a^4, [c, b] = c^2 \rangle$ ; where  $|G| = 2^6$ ,  $\Phi(G) = \langle a^2, c \rangle \cong C_4^2$ ,  $G' = \langle c, a^4 \rangle$  and  $Z(G) = \langle a^4, c^2 \rangle \cong C_2^2$ .
- (C17)  $\langle a, b, c \mid a^{p^3} = b^{p^2} = c^p = 1, [a, b] = c, [c, a] = b^{\nu_1 p}, [c, b] = a^{-\nu_2 p^2} \rangle$ , where  $p \geq 3$ , and  $\nu_1, \nu_2 = 1$  or a fixed quadratic non-residue modulo  $p$ ; moreover,  $|G| = p^6$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^2} \times C_p \times C_p$ ,  $G' = \langle c, a^{p^2}, b^p \rangle$  and  $Z(G) = \langle a^p, b^p \rangle \cong C_{p^2} \times C_p$ .

Moreover, Table 4 gives  $(\mu_0, \mu_1, \mu_2)$  and  $\alpha_1(G)$  for the groups (C7)–(C17).

$(\mu_0, \mu_1, \mu_2)$	$\alpha_1(G)$	types of $\mathcal{A}_3$ groups
$(0, p-1, 2)$	$2p^2 + p - 1$	(C7); (C10); (C12)–(C14)
$(0, p, 1)$	$p^2 + p$	(C8)–(C9); (C11); (C15); (C17)
$(0, p, 1)$	$p^2 + 2p$	(C16)

Table 4: The enumeration of (C7)–(C17)

**Proof** Let  $A$  and  $B$  be two distinct  $\mathcal{A}_1$ -subgroups of index  $p$ . Then  $|A'| = |B'| = p$  and hence  $A'B' \leq Z(G)$ . Let  $\bar{G} = G/A'B'$ . Then  $\bar{G}$  has two abelian subgroups  $\bar{A}$  and  $\bar{B}$  of index  $p$ . By Lemma 2.7,  $\bar{G}$  has  $1 + p$  abelian subgroups. Since  $d(\bar{G}) = 2$ ,  $\bar{G}$  is an  $\mathcal{A}_1$ -group. It is easy to see that  $A'B' = \Phi(G')G_3$ . If  $\bar{G}$  is metacyclic, then  $G$  is also metacyclic by Lemma 2.1 and hence  $A' = B'$ . It follows that  $|G'| = p^2$ . by Lemma 3.1,  $G$  is an  $\mathcal{A}_2$ -group. This contradicts that  $G$  is an  $\mathcal{A}_3$ -group. Thus  $\bar{G}$  is a non-metacyclic  $\mathcal{A}_1$ -group.

If  $G' \cong C_{p^2}$ , then  $G_3 \leq \Phi(G') \cong C_p$ . Thus  $G$  is one of the groups listed in [3, Theorem 3.5]. By [3, Theorem 3.6] we get no group.

If  $G' \cong C_p^2$ , then  $G$  is one of the groups listed in [3, Theorem 4.6]. By [3, Theorem 4.7] we get groups (C1)–(C6).

If  $G_3 \cong C_p$  and  $\Phi(G')G_3 \cong C_p^2$ , then, by [3, Theorem 5.1, 5.2, 5.5 & 5.6], we get no group.

If  $\Phi(G') \leq G_3 \cong C_p^2$ , then  $G$  is one of the groups listed in [3, Theorem 6.5]. By [3, Theorem 6.1], we get the groups (C7)–(C17).

Table 5 gives the correspondence from Theorem 4.4 to [3, Theorem 4.6 & 6.5].

Groups	Groups in [3, Theorem 4.6]	Groups	Groups in [3, Theorem 6.5]
(C1)	(H1)	(C7)	(O1)
(C2)	(H2)	(C8)	(O3)
(C3)	(H3)	(C9)	(O4)
(C4)	(I1)	(C10)	(O5)
(C5)	(I2)	(C11)	(P2) where $n = 1$
(C6)	(I3)	(C12)	(P3) where $n = 1$ and $-\nu \in F_p^2$
		(C13)	(P4) where $n = 1$ and $-r \in (F_p^*)^2$
		(C14)	(P8) where $n = 1$
		(C15)	(P9) where $n = 1$
		(C16)	(R4)
		(C17)	(S2) where $n = 2$

Table 5: The correspondence from Theorem 4.4 to [3, Theorem 4.6 & 6.5]

We calculate the  $(\mu_0, \mu_1, \mu_2)$  and  $\alpha_1(G)$  of those groups in Theorem 4.4 as follows.

Since  $d(G) = 2$ ,  $G$  has  $1 + p$  maximal subgroups. For any  $\mathcal{A}_3$ -group, we always have  $\mu_2 \geq 1$ . By the hypothesis of Theorem 4.4,  $\mu_0 = 0$  and  $\mu_1 \geq 2$ .

**Case 1.**  $c(G) = 3$  and  $G' \cong C_2^2$ . That is,  $G$  is one of the groups (C1)–(C6).

Obviously,  $(\mu_0, \mu_1, \mu_2) = (0, 2, 1)$ . Let  $M \in \Gamma_1$ . If  $M$  is the unique  $\mathcal{A}_2$ -subgroup, then  $|M'| = p$ . By Lemma 2.6,  $\alpha_1(M) = p^2 = 4$ . By Hall's enumeration principle,  $\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = \mu_1 + 4\mu_2 = 6$ .

**Case 2.**  $G$  is one of the groups (C7)–(C15).

In this case,  $p \geq 3$ ,  $\Phi(G') = 1$ ,  $G_3 \cong C_p^2$ ,  $G_3 \leq Z(G)$  and  $G/G_3 \cong M_p(1, 1, 1)$ .  $G = \langle a, b, c; x, y \rangle$  such that

$$a^p = x^{w_{11}} y^{w_{12}}, b^p = x^{w_{21}} y^{w_{22}}, c^p = x^p = y^p = 1, [a, b] = c, [c, a] = x, [c, b] = y.$$

Let  $N = \langle c, b, x, y \rangle$  and  $M_i = \langle ab^i, c, x, y \rangle$ . Then  $N$  and  $M_i$  are all maximal subgroups of  $G$ . Let  $w(G) = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$ .

If  $G$  is the group (C7), then  $p = 3$  and  $w(G) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ . By Lemma 4.3,  $N \in \mathcal{A}_2$  and  $M_i \in \mathcal{A}_2$  if and only if  $i = 0$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p - 1, 2)$  and  $\alpha_1(G) = 2p^2 + p - 1$ .

If  $G$  is the group (C8), then  $p = 3$  and  $w(G) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ . By Lemma 4.3,  $N \in \mathcal{A}_2$  and  $M_i \in \mathcal{A}_1$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p, 1)$  and  $\alpha_1(G) = p^2 + p$ .

If  $G$  is the group (C9), then  $p = 3$  and  $w(G) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$ . By Lemma 4.3,  $N \in \mathcal{A}_2$  and  $M_i \in \mathcal{A}_1$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p, 1)$  and  $\alpha_1(G) = p^2 + p$ .



If  $G$  is the group (C10), then  $p = 3$  and  $w(G) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . By Lemma 4.3,  $N \in \mathcal{A}_2$  and  $M_i \in \mathcal{A}_2$  if and only if  $i = 0$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p - 1, 2)$  and  $\alpha_1(G) = 2p^2 + p - 1$ .

If  $G$  is the group (C11), then  $p > 3$  and  $w(G) = \begin{pmatrix} 1 & -\nu \\ 0 & 1 \end{pmatrix}$ . By Lemma 4.3,  $N \in \mathcal{A}_2$  and  $M_i \in \mathcal{A}_1$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p, 1)$  and  $\alpha_1(G) = p^2 + p$ .

If  $G$  is the group (C12), then  $p > 3$  and  $w(G) = \begin{pmatrix} 0 & -1 \\ \nu^{-1} & 0 \end{pmatrix}$ . By Lemma 4.3,  $N \in \mathcal{A}_1$  and  $M_i \in \mathcal{A}_2$  if and only if  $i^2 = -\nu$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p - 1, 2)$  and  $\alpha_1(G) = 2p^2 + p - 1$ .

If  $G$  is the group (C13), then  $p > 3$  and  $w(G) = \begin{pmatrix} 1 & -1 \\ r & 1 \end{pmatrix}$ . By Lemma 4.3,  $N \in \mathcal{A}_1$  and  $M_i \in \mathcal{A}_2$  if and only if  $i^2 = -r^{-1}$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p - 1, 2)$  and  $\alpha_1(G) = 2p^2 + p - 1$ .

If  $G$  is the group (C14), then  $p > 3$  and  $w(G) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ . By Lemma 4.3,  $N \in \mathcal{A}_2$  and  $M_i \in \mathcal{A}_2$  if and only if  $i = 0$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p - 1, 2)$  and  $\alpha_1(G) = 2p^2 + p - 1$ .

If  $G$  is the group (C15), then  $p > 3$  and  $w(G) = \begin{pmatrix} 0 & 0 \\ \nu^{-1} & 0 \end{pmatrix}$ . By Lemma 4.3,  $N \in \mathcal{A}_1$  and  $M_i \in \mathcal{A}_2$  if and only if  $i = 0$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p, 1)$  and  $\alpha_1(G) = p^2 + p$ .

**Case 3.**  $G$  is the group (C16).

It is obvious that  $(\mu_0, \mu_1, \mu_2) = (0, 2, 1) = (0, p, 1)$ . The unique  $\mathcal{A}_2$ -subgroup of index  $p$  is  $M = \langle c, b, a^2 \rangle$ , where  $d(M) = 3$  and  $|M'| = 4$ . Notice that  $\alpha_1(M) = p^2 + p = 6$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = \mu_1 + (p^2 + p)\mu_2 = p^2 + 2p = 8.$$

**Case 4.**  $G$  is the group (C17).

Let  $N = \langle c, b, a^p, b^p \rangle$  and  $M_i = \langle c, ab^i, a^p, b^p \rangle$ . Then  $N$  and  $M_i$  are all maximal subgroups of  $G$ . Since  $N = \langle c, b \rangle * \langle a^p \rangle$ ,  $N \in \mathcal{A}_2$  with  $d(N) = 3$  and  $|N'| = p$ . By Lemma 2.6 (7),  $\alpha_1(N) = p^2$ . Since  $M_i = \langle c, ab^i \rangle$  and  $|M'_i| = p$ ,  $M_i \in \mathcal{A}_1$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p, 1)$  and, by Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = \mu_1 + p^2\mu_2 = p^2 + p.$$

To sum up, we get the Table 4. □

**Lemma 4.5.** Suppose that  $G$  is an  $\mathcal{A}_3$ -group with  $d(G) = 3$ ,  $\Phi(G) = Z(G)$  and  $G' \cong C_p^3$ . Then we may assume that  $G = \langle a_1, a_2, a_3; x, y, z \rangle$  with

$$x = [a_2, a_3], y = [a_3, a_1], z = [a_1, a_2], x^p = y^p = z^p = 1, a_i^{p^{m_i}} = x^{w_{i1}} y^{w_{i2}} z^{w_{i3}}, i = 1, 2, 3.$$

Let  $M = \langle a_2, a_3, \Phi(G) \rangle$ ,  $M_i = \langle a_1 a_2^i, a_3, \Phi(G) \rangle$  and  $M_{ij} = \langle a_1 a_3^i, a_2 a_3^j, \Phi(G) \rangle$ , where  $0 \leq i, j \leq p-1$ . Then  $M$ ,  $M_i$  and  $M_{ij}$  are all maximal subgroups of  $G$ , and the following conclusions hold:

(1) If  $m_1 = 2$ ,  $m_2 = m_3 = 1$  and  $a_1^{p^2} = x$ , then  $M \in \mathcal{A}_2$ ,  $M_i \in \mathcal{A}_2$  if and only if  $w_{33} = 0$ , and  $M_{ij} \in \mathcal{A}_2$  if and only if  $w_{22} + j(w_{23} + w_{32}) + j^2 w_{33} = 0$ ;

(2) If  $m_1 = m_2 = 2$ ,  $m_3 = 1$ ,  $a_3^p = z$  and  $w_{13} = w_{23} = 0$ , then  $M \in \mathcal{A}_2$ ,  $M_i \in \mathcal{A}_2$ , and  $M_{ij} \in \mathcal{A}_2$  if and only if  $w_{11}w_{22} - w_{12}w_{21} = 0$ ;

(3) If  $m_1 = m_2 = m_3 = 1$  and  $p > 2$ , then  $M \in \mathcal{A}_2$  if and only if  $\begin{vmatrix} w_{22} & w_{23} \\ w_{32} & w_{33} \end{vmatrix} = 0$ ,  $M_i \in \mathcal{A}_2$  if and only if  $\begin{vmatrix} i & -1 & 0 \\ w_{11} + iw_{21} & w_{12} + iw_{22} & w_{13} + iw_{23} \\ w_{31} & w_{32} & w_{33} \end{vmatrix} = 0$ , and  $M_{ij} \in \mathcal{A}_2$  if and only if  $\begin{vmatrix} -i & -j & 1 \\ w_{11} + iw_{31} & w_{12} + iw_{32} & w_{13} + iw_{33} \\ w_{21} + jw_{31} & w_{22} + jw_{32} & w_{23} + jw_{33} \end{vmatrix} = 0$ ;

(4) If  $m_1 = m_2 = m_3 = 1$  and  $p = 2$ , then  $M \in \mathcal{A}_2$  if and only if  $\begin{vmatrix} w_{22} & w_{23} \\ w_{32} & w_{33} \end{vmatrix} = 0$ ,  $M_i \in \mathcal{A}_2$  if and only if  $\begin{vmatrix} i & 1 & 0 \\ w_{11} + iw_{21} & w_{12} + iw_{22} & w_{13} + iw_{23} + i \\ w_{31} & w_{32} & w_{33} \end{vmatrix} = 0$ , and  $M_{ij} \in \mathcal{A}_2$  if and only if  $\begin{vmatrix} i & j & 1 \\ w_{11} + iw_{31} & w_{12} + iw_{32} + i & w_{13} + iw_{33} \\ w_{21} + jw_{31} + j & w_{22} + jw_{32} & w_{23} + jw_{33} \end{vmatrix} = 0$ .

**Proof** Since  $G \in \mathcal{A}_3$ , any maximal subgroup is either  $\mathcal{A}_1$ -group or  $\mathcal{A}_2$ -group. Let  $H \in \Gamma_1(G)$ . Then  $|H'| = p$ . If  $d(H) = 2$ , then, by Lemma 2.2,  $H \in \mathcal{A}_1$ . Since  $\Phi(H) \leq \Phi(G)$ , we get that  $H \in \mathcal{A}_1$  if and only if  $\Phi(G) = \Phi(H)$ .

(1) If  $m_1 = 2$ ,  $m_2 = m_3 = 1$  and  $a_1^{p^2} = x$ , then it is obvious that  $M = \langle a_2, a_3 \rangle * \langle a_1 \rangle \in \mathcal{A}_2$ . Since  $x = (a_1 a_2^i)^{p^2} \in \Phi(M_i)$  and  $y = [a_3, a_1 a_2^i] x^i \in \Phi(M_i)$ ,  $\Phi(M_i) = \Phi(G)$  if and only if  $a_3^p \notin \langle x, y \rangle$ . Hence  $M_i \in \mathcal{A}_2$  if and only if  $w_{33} = 0$ . Similar reason gives that  $M_{ij} \in \mathcal{A}_1$  if and only if  $(a_2 a_3^j)^p \notin \langle x, [a_1 a_3^i, a_2 a_3^j] \rangle = \langle x, y^{-j} z \rangle$ . Hence  $M_{ij} \in \mathcal{A}_2$  if and only if  $\begin{vmatrix} 1 & 0 & 0 \\ 0 & -j & 1 \\ * & w_{22} + jw_{32} & w_{23} + jw_{33} \end{vmatrix} = 0$ . The later is,  $w_{22} + j(w_{23} + w_{32}) + j^2 w_{33} = 0$ .

(2) If  $m_1 = m_2 = 2$ ,  $m_3 = 1$ ,  $a_3^p = z$  and  $w_{13} = w_{23} = 0$ , then it is obvious that  $a_1^p \notin \langle a_2, a_3 \rangle$  and  $\langle a_1^p, a_2^p \rangle \not\leq \langle a_1 a_2^i, a_3 \rangle$ . Hence  $M \in \mathcal{A}_2$  and  $M_i \in \mathcal{A}_2$ . Since  $(a_1 a_3^i)^{p^2} = a_1^{p^2} = x^{w_{11}} y^{w_{12}}$  and  $(a_2 a_3^j)^{p^2} = a_2^{p^2} = x^{w_{21}} y^{w_{22}}$  and  $[a_1 a_2^i, a_2 a_3^j] \notin \langle x, y \rangle$   $M_{ij} \in \mathcal{A}_1$  if and only if  $\begin{vmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{vmatrix} = 0$ . The later is,  $w_{11}w_{22} - w_{12}w_{21} = 0$ .

(3) If  $m_1 = m_2 = m_3 = 1$  and  $p > 2$ , then  $M \in \mathcal{A}_1$  if and only if  $\langle a_2^p, a_3^p, [a_2, a_3] \rangle = G'$ . The later is  $\begin{vmatrix} w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \\ 1 & 0 & 0 \end{vmatrix} \neq 0$ . Hence  $M \in \mathcal{A}_2$  if and only if  $\begin{vmatrix} w_{22} & w_{23} \\ w_{32} & w_{33} \end{vmatrix} = 0$ .

Similar arguments give that  $M_i \in \mathcal{A}_2$  if and only if  $\begin{vmatrix} i & -1 & 0 \\ w_{11} + iw_{21} & w_{12} + iw_{22} & w_{13} + iw_{23} \\ w_{31} & w_{32} & w_{33} \end{vmatrix} = 0$ , and  $M_{ij} \in \mathcal{A}_2$  if and only if  $\begin{vmatrix} -i & -j & 1 \\ w_{11} + iw_{31} & w_{12} + iw_{32} & w_{13} + iw_{33} \\ w_{21} + jw_{31} & w_{22} + jw_{32} & w_{23} + jw_{33} \end{vmatrix} = 0$ .

(4) It follows from similar arguments as (3). The details are omitted.  $\square$

**Theorem 4.6.**  *$G$  has no abelian subgroup of index  $p$ ,  $G$  has at least two distinct  $\mathcal{A}_1$ -subgroups of index  $p$  and  $d(G) = 3$  if and only if  $G$  is isomorphic to one of the following pairwise non-isomorphic groups:*

(Di) *The type of  $G/G'$  is  $(p^2, p, p)$ .*

- (D1)  $\langle a, b, c \mid a^{p^3} = b^{p^2} = c^{p^2} = 1, [b, c] = a^{p^2}, [c, a] = c^{-p}, [a, b] = b^p c^{\nu p} \rangle$ , where  $p > 2$  and  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ ; moreover,  $|G| = p^7$ ,  $\Phi(G) = Z(G) = \langle a^p, b^p, c^p \rangle \cong C_{p^2} \times C_p \times C_p$  and  $G' = \langle a^{p^2}, b^p, c^p \rangle \cong C_p^3$ .
- (D2)  $\langle a, b, c \mid a^{p^3} = b^{p^2} = c^{p^2} = 1, [b, c] = a^{p^2}, [c, a] = b^p, [a, b] = c^{\nu p} \rangle$ , where  $p > 2$ ,  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ ; moreover,  $|G| = p^7$ ,  $\Phi(G) = Z(G) = \langle a^p, b^p, c^p \rangle \cong C_{p^2} \times C_p \times C_p$  and  $G' = \langle a^{p^2}, b^p, c^p \rangle \cong C_p^3$ .
- (D3)  $\langle a, b, c \mid a^{p^3} = b^{p^2} = c^{p^2} = 1, [b, c] = a^{p^2}, [c, a]^{1+r} = b^{rp} c^{-p}, [a, b]^{1+r} = b^p c^p \rangle$ , where  $p > 2$ ,  $r = 1, 2, \dots, p-2$ ; moreover,  $|G| = p^7$ ,  $\Phi(G) = Z(G) = \langle a^p, b^p, c^p \rangle \cong C_{p^2} \times C_p \times C_p$  and  $G' = \langle a^{p^2}, b^p, c^p \rangle \cong C_p^3$ .
- (D4)  $\langle a, b, c \mid a^8 = b^4 = c^4 = 1, [b, c] = a^4, [c, a] = b^2, [a, b] = c^2 \rangle$ ; where  $|G| = 2^7$ ,  $\Phi(G) = Z(G) = \langle a^2, b^2, c^2 \rangle \cong C_4 \times C_2 \times C_2$  and  $G' = \langle a^4, b^2, c^2 \rangle \cong C_2^3$ .
- (D5)  $\langle a, b, c \mid a^8 = b^4 = c^4 = 1, [b, c] = a^4, [c, a] = b^2, [a, b] = b^2 c^2 \rangle$ , where  $|G| = 2^7$ ,  $\Phi(G) = Z(G) = \langle a^2, b^2, c^2 \rangle \cong C_4 \times C_2 \times C_2$  and  $G' = \langle a^4, b^2, c^2 \rangle \cong C_2^3$ .

(Dii) *The type of  $G/G'$  is  $(p^2, p^2, p)$ .*

- (D6)  $\langle a, b, c \mid a^{p^3} = b^{p^3} = c^{p^2} = 1, [b, c] = a^{p^2}, [c, a] = b^{\nu p^2}, [a, b] = c^p \rangle$ , where  $p > 2$ ,  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$  such that  $-\nu \notin (F_p)^2$ ; moreover,  $|G| = p^8$ ,  $\Phi(G) = Z(G) = \langle a^p, b^p, c^p \rangle \cong C_{p^2} \times C_{p^2} \times C_p$  and  $G' = \langle a^{p^2}, b^{p^2}, c^p \rangle \cong C_p^3$ .
- (D7)  $\langle a, b, c \mid a^{p^3} = b^{p^3} = c^{p^2} = 1, [b, c]^{1+r} = a^{rp^2} b^{p^2}, [c, a]^{1+r} = a^{-p^2} b^{p^2}, [a, b] = c^p \rangle$ , where  $p > 2$ ,  $r = 1, 2, \dots, p-2$  such that  $-r \notin (F_p)^2$ ; moreover,  $|G| = p^8$ ,  $\Phi(G) = Z(G) = \langle a^p, b^p, c^p \rangle \cong C_{p^2} \times C_{p^2} \times C_p$  and  $G' = \langle a^{p^2}, b^{p^2}, c^p \rangle \cong C_p^3$ .
- (D8)  $\langle a, b, c \mid a^8 = b^8 = c^4 = 1, [b, c] = a^4 b^4, [c, a] = b^4, [a, b] = c^2 \rangle$ ; where  $|G| = 2^8$ ,  $\Phi(G) = Z(G) = \langle a^2, b^2, c^2 \rangle \cong C_4 \times C_4 \times C_2$  and  $G' = \langle a^4, b^4, c^2 \rangle \cong C_2^3$ .

(Diii) *The type of  $G/G'$  is  $(p, p, p)$  where  $p > 2$ .*

- (D9)  $\langle a, b, c \mid a^{p^2} = b^{p^2} = c^{p^2} = 1, [b, c] = a^p, [c, a] = b^p, [a, b] = c^p \rangle$ ; where  $|G| = p^6$ ,  $\Phi(G) = Z(G) = G' = \langle a^p, b^p, c^p \rangle \cong C_p^3$ .
- (D10)  $\langle a, b, c \mid a^{p^2} = b^{p^2} = c^{p^2} = 1, [b, c] = a^p, [c, a] = c^{-p}, [a, b] = b^p \rangle$ ; where  $|G| = p^6$ ,  $\Phi(G) = Z(G) = G' = \langle a^p, b^p, c^p \rangle \cong C_p^3$ .
- (D11)  $\langle a, b, c \mid a^{p^2} = b^{p^2} = c^{p^2} = 1, [b, c] = a^p, [c, a] = c^{-p}, [a, b] = b^p c^{\nu p} \rangle$ , where  $\nu = 1$  or is a fixed quadratic non-residue modulo  $p$ ; moreover,  $|G| = p^6$ ,  $\Phi(G) = Z(G) = G' = \langle a^p, b^p, c^p \rangle \cong C_p^3$ .

- (D12)  $\langle a, b, c \mid a^{p^2} = b^{p^2} = c^{p^2} = 1, [b, c] = a^p, [c, a]^{1+r} = b^{rp}c^{-p}, [a, b]^{1+r} = b^p c^p \rangle$ ,  
where  $r = 1, 2, \dots, p-2$ . Moreover,  $|G| = p^6$ ,  $\Phi(G) = Z(G) = G' = \langle a^p, b^p, c^p \rangle \cong C_p^3$ .
- (D13)  $\langle a, b, c \mid a^{p^2} = b^{p^2} = c^{p^2} = 1, [b, c] = a^{-p}b^p c^p, [c, a] = a^{-p}b^p, [a, b] = a^p \rangle$ ;  
where  $|G| = p^6$ ,  $\Phi(G) = Z(G) = G' = \langle a^p, b^p, c^p \rangle \cong C_p^3$ .
- (D14)  $\langle a, b, c; d \mid a^{p^2} = b^{p^2} = c^p = d^p = 1, [b, c] = a^p, [c, a] = b^{\nu p}, [a, b] = d, [d, a] = [d, b] = [d, c] = 1 \rangle$ , where  $p > 2$ ,  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$  such that  $-\nu \notin (F_p)^2$ ; moreover,  $|G| = p^6$ ,  $\Phi(G) = Z(G) = G' = \langle a^p, b^p, d \rangle \cong C_p^3$ .
- (D15)  $\langle a, b, c; d \mid a^{p^2} = b^{p^2} = c^p = d^p = 1, [b, c] = a^p, [c, a] = d, [a, b] = b^p, [d, a] = [d, b] = [d, c] = 1 \rangle$ , where  $p > 2$ ; moreover,  $|G| = p^6$ ,  $\Phi(G) = Z(G) = G' = \langle a^p, b^p, d \rangle \cong C_p^3$ .
- (D16)  $\langle a, b, c; d \mid a^p = b^{p^2} = c^{p^2} = d^p = 1, [b, c] = d, [c, a]^{1+r} = b^{rp}c^{-p}, [a, b]^{1+r} = b^p c^p, [d, a] = [d, b] = [d, c] = 1 \rangle$ , where  $p > 2$ ,  $r = 1, 2, \dots, p-2$  such that  $-r \notin (F_p)^2$ ; moreover,  $|G| = p^6$ ,  $\Phi(G) = Z(G) = G' = \langle b^p, c^p, d \rangle \cong C_p^3$ .

(Div) The type of  $G/G'$  is  $(2, 2, 2)$ .

- (D17)  $\langle a, b, c; d \mid a^4 = b^2 = c^4 = d^2 = 1, [b, c] = d, [c, a] = a^2, [a, b] = c^2, [d, a] = [d, b] = [d, c] = 1 \rangle$ ; where  $\Phi(G) = Z(G) = G' = \langle a^2, c^2, d \rangle \cong C_2^3$ .
- (D18)  $\langle a, b, c; d \mid a^4 = b^4 = c^4 = d^2 = 1, [b, c] = d, [c, a] = a^2, [a, b] = b^2 = c^2, [d, a] = [d, b] = [d, c] = 1 \rangle$ ; where  $\Phi(G) = Z(G) = G' = \langle a^2, b^2, d \rangle \cong C_2^3$ .
- (D19)  $\langle a, b, c; d \mid a^4 = b^4 = c^4 = d^2 = 1, [b, c] = d, [c, a] = a^2 b^2, [a, b] = a^2 = c^2, [d, a] = [d, b] = [d, c] = 1 \rangle$ ;  $\Phi(G) = Z(G) = G' = \langle a^2, b^2, d \rangle \cong C_2^3$ .

Moreover, Table 6 gives  $(\mu_0, \mu_1, \mu_2)$  and  $\alpha_1(G)$  for the groups (D1)–(D19).

$(\mu_0, \mu_1, \mu_2)$	$\alpha_1(G)$	types of $\mathcal{A}_3$ groups
$(0, p^2, p+1)$	$p^3 + 2p^2$	(D1), (D4), (D6)–(D10), (D12)–(D14), (D16)
$(0, p^2 + p, 1)$	$2p^2 + p$	(D2) where $-\nu \notin (F_p^*)^2$ ; (D3) where $-r \notin (F_p^*)^2$ ; (D5); (D11) where $-\nu \notin (F_p^*)^2$ ;
$(0, p^2 - p, 2p+1)$	$2p^3 + 2p^2 - p$	(D2) where $-\nu \in (F_p^*)^2$ ; (D3) where $-\nu \in (F_p^*)^2$ ; (D11) where $-\nu \in (F_p^*)^2$ ; (D15)
$(0, p^2 - 1, p+2)$	$p^3 + 3p^2 - 1$	(D17), (D18)
$(0, p^2 + 1, p)$	$p^3 + p^2 + 1$	(D19)

Table 6: The enumeration of (D1)–(D19)

**Proof** Let  $A$  and  $B$  be two distinct  $\mathcal{A}_1$ -subgroups of index  $p$ . By Lemma 2.15,  $|G'| \leq p|A'||B'| \leq p^3$ . It is easy to see that  $\Phi(A) = \Phi(B) = \Phi(G)$ . By Lemma 2.2, we have  $\Phi(A) = Z(A)$  and  $\Phi(B) = Z(B)$ . Since  $[\Phi(G), A] = [\Phi(G), B] = 1$  and  $G = AB$ , we have  $\Phi(G) \leq Z(G)$ . Moreover,  $G' \leq C_p^3$ . If  $|G'| \leq p^2$ , then  $G$  has an abelian subgroup of index  $p$ , a contradiction. Now we have  $d(G) = 3$ ,  $\Phi(G) \leq Z(G)$  and  $G' \cong C_p^3$ . Thus  $G$  is one of the groups classified in [23]. Let the type of  $G/G'$  be  $(p^{m_1}, p^{m_2}, p^{m_3})$ , where  $m_1 \geq m_2 \geq m_3$ . By [23, Theorem 2.7],  $m_3 = 1$ . By [23, Theorem 2.8],  $m_1 \leq 2$ .

If  $m_1 = 2$  and  $m_2 = 1$ , then  $G$  is either one of the groups determined by [23, Theorem 4.1] for  $p > 2$  or one of the groups determined by [23, Theorem 7.1] for  $p = 2$ . By [23, Theorem 4.3 & 7.4], we get the groups (D1)–(D5).

If  $m_1 = m_2 = 2$ , then  $G$  is one of the groups determined by [23, Theorem 5.1]. By [23, Theorem 5.2], we get the groups (D6)–(D8).

If  $m_1 = 1$  and  $p > 2$ , then  $G$  is one of the groups determined by [23, Theorem 6.1]. By [23, Theorem 6.3], we get the groups (D9)–(D16).

If  $m_1 = 1$  and  $p = 2$ , then  $G$  is one of the groups listed in [23, Theorem 7.6]. By [23, Theorem 7.7], we get the groups (D17)–(D19). Table 7 gives the correspondence.

Groups	Groups in [23, Theorem 4.1 & 5.1 & 6.1 & 7.1 & 7.6]	Groups	Groups in [23, Theorem 4.1 & 5.1 & 6.1 & 7.1 & 7.6]
(D1)	(D2) where $m_1 = 2$ and $m_2 = 1$	(D11)	(J3) where $m_1 = m_2 = m_3 = 1$
(D2)	(D3) where $m_1 = 2$ and $m_2 = 1$	(D12)	(J4) where $m_1 = m_2 = m_3 = 1$
(D3)	(D4) where $m_1 = 2$ and $m_2 = 1$	(D13)	(J5) where $m_1 = m_2 = m_3 = 1$
(D4)	(M2) where $m_1 = 2$	(D14)	(K1) where $m_1 = m_2 = m_3 = 1$ and $-\nu \notin F_p^2$
(D5)	(M3) where $m_1 = 2$	(D15)	(K2)
(D6)	(G3) where $-\nu \notin F_p^2$ , $m_1 = m_2 = 2$ and $m_3 = 1$	(D16)	(K6) where $m_1 = m_2 = m_3 = 1$ and $-r \notin F_p^2$
(D7)	(G4) where $-r \notin F_p^2$ , $m_1 = m_2 = 2$ and $m_3 = 1$	(D17)	(S7)
(D8)	(G7) where $m_1 = m_2 = 2$ and $m_3 = 1$	(D18)	(S8)
(D9)	(J1) where $m_1 = m_2 = m_3 = 1$	(D19)	(S9)
(D10)	(J2) where $m_1 = m_2 = m_3 = 1$		

Table 7: The correspondence from Theorem 4.6 to [23, Theorem 4.1, 5.1, 6.1, 7.1 & 7.6]

We calculate the  $(\mu_0, \mu_1, \mu_2)$  and  $\alpha_1(G)$  of those groups in Theorem 4.6 as follows.

By Lemma 3.18,  $\alpha_1(G) = \mu_1 + p^2\mu_2$ . Hence we only need to calculate  $(\mu_0, \mu_1, \mu_2)$ . we may assume that  $G = \langle a_1, a_2, a_3; x, y, z \rangle$  with

$$x = [a_2, a_3], y = [a_3, a_1], z = [a_1, a_2], x^p = y^p = z^p = 1, a_i^{p^{m_i}} = x^{w_{i1}} y^{w_{i2}} z^{w_{i3}}, i = 1, 2, 3.$$

Let  $M = \langle a_2, a_3, \Phi(G) \rangle$ ,  $M_i = \langle a_1 a_2^i, a_3, \Phi(G) \rangle$  and  $M_{ij} = \langle a_1 a_3^i, a_2 a_3^j, \Phi(G) \rangle$ , where  $0 \leq i, j \leq p-1$ . Then  $M$ ,  $M_i$  and  $M_{ij}$  are all maximal subgroups of  $G$ . Let  $w(G) = (w_{ij})$ .

If  $G$  is the group (D1), then  $m_1 = 2$ ,  $m_2 = m_3 = 1$  and  $w(G) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \nu & 1 \\ 0 & -1 & 0 \end{pmatrix}$ . By Lemma 4.5 (1),  $M \in \mathcal{A}_2$ ,  $M_i \in \mathcal{A}_2$  and  $M_{ij} \in \mathcal{A}_1$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p^2, p+1)$  and  $\alpha_1(G) = p^3 + 2p^2$ .

If  $G$  is the group (D2), then  $m_1 = 2$ ,  $m_2 = m_3 = 1$  and  $w(G) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu^{-1} \end{pmatrix}$ . By Lemma 4.5 (1),  $M \in \mathcal{A}_2$ ,  $M_i \in \mathcal{A}_1$  and  $M_{ij} \in \mathcal{A}_2$  if and only if  $j^2 = -\nu$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p^2 + p, 1)$  for  $-\nu \notin (F_p^*)^2$  and  $(\mu_0, \mu_1, \mu_2) = (0, p^2 - p, 2p + 1)$  for  $-\nu \in (F_p^*)^2$ .

If  $G$  is the group (D3), then  $m_1 = 2$ ,  $m_2 = m_3 = 1$  and  $w(G) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & r \end{pmatrix}$ . By Lemma 4.5 (1),  $M \in \mathcal{A}_2$ ,  $M_i \in \mathcal{A}_1$  and  $M_{ij} \in \mathcal{A}_2$  if and only if  $j^2 = -r^{-1}$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p^2 + p, 1)$  for  $-r \notin (F_p^*)^2$  and  $(\mu_0, \mu_1, \mu_2) = (0, p^2 - p, 2p + 1)$  for  $-r \in (F_p^*)^2$ .

If  $G$  is the group (D4), then  $m_1 = 2$ ,  $m_2 = m_3 = 1$  and  $w(G) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ . By Lemma 4.5 (1),  $M \in \mathcal{A}_2$ ,  $M_i \in \mathcal{A}_1$  and  $M_{ij} \in \mathcal{A}_2$  if and only if  $j = 1$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p^2, p+1)$  and  $\alpha_1(G) = p^3 + 2p^2$ .

If  $G$  is the group (D5), then  $m_1 = 2$ ,  $m_2 = m_3 = 1$  and  $w(G) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ . By Lemma 4.5 (1),  $M \in \mathcal{A}_2$ ,  $M_i \in \mathcal{A}_1$  and  $M_{ij} \in \mathcal{A}_1$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p^2 + p, 1)$  and  $\alpha_1(G) = 2p^2 + p$ .

If  $G$  is one of the groups (D6)–(D8), then  $m_1 = m_2 = 2$ ,  $m_3 = 1$  and  $w(G) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ,  $\begin{pmatrix} 1 & -1 & 0 \\ 1 & r & 0 \\ 0 & 0 & 1 \end{pmatrix}$ , or  $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ . By Lemma 4.5 (2),  $M \in \mathcal{A}_2$ ,  $M_i \in \mathcal{A}_2$  and  $M_{ij} \in \mathcal{A}_1$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p^2, p+1)$  and  $\alpha_1(G) = p^3 + 2p^2$ .

If  $G$  is the group (D9), then  $p > 2$  and  $m_1 = m_2 = m_3 = 1$  and  $w(G) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ . By Theorem 4.5 (3),  $M \in \mathcal{A}_1$ ,  $M_i \in \mathcal{A}_2$  if and only if  $i^2 + 1 = 0$  and  $M_{ij} \in \mathcal{A}_2$  if and only if  $i^2 + j^2 + 1 = 0$ .

If  $-1 \in (F_p^*)^2$ , then  $i^2 + 1 = 0$  has two solutions, and by Lemma 2.9,  $i^2 + j^2 + 1 = 0$  has  $p - 1$  solutions. Hence  $(\mu_0, \mu_1, \mu_2) = (0, p^2, p+1)$ . If  $-1 \notin (F_p^*)^2$ , then  $i^2 + 1 = 0$  has no solution, and by Lemma 2.9,  $i^2 + j^2 + 1 = 0$  has  $p + 1$  solutions. Hence we also have  $(\mu_0, \mu_1, \mu_2) = (0, p^2, p+1)$ .

If  $G$  is the group (D10), then  $p > 2$  and  $m_1 = m_2 = m_3 = 1$  and  $w(G) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ . By Theorem 4.5 (3),  $M \in \mathcal{A}_1$ ,  $M_i \in \mathcal{A}_2$  if and only if  $i = 0$  and  $M_{ij} \in \mathcal{A}_2$  if and only if  $i = 0$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p^2, p+1)$ .

If  $G$  is the group (D11), then  $p > 2$  and  $m_1 = m_2 = m_3 = 1$  and  $w(G) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \nu & 1 \\ 0 & -1 & 0 \end{pmatrix}$ . By Theorem 4.5 (3),  $M \in \mathcal{A}_1$ ,  $M_i \in \mathcal{A}_2$  if and only if  $i = 0$  and  $M_{ij} \in \mathcal{A}_2$  if and only if  $i^2 = -\nu$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p^2 + p, 1)$  for  $-\nu \notin (F_p^*)^2$  and  $(\mu_0, \mu_1, \mu_2) = (0, p^2 - p, 2p + 1)$  for  $-\nu \in (F_p^*)^2$ .

If  $G$  is the group (D12), then  $p > 2$  and  $m_1 = m_2 = m_3 = 1$  and  $w(G) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & r \end{pmatrix}$ . By Theorem 4.5 (3),  $M \in \mathcal{A}_1$ ,  $M_i \in \mathcal{A}_2$  if and only if  $i^2(r+1) + r = 0$  and  $M_{ij} \in \mathcal{A}_2$  if and only if  $i^2(r+1) + rj^2 + 1 = 0$ .

If  $-(r+1)^{-1}r \in (F_p^*)^2$ , then  $i^2(r+1) + r = 0$  has two solutions, and by Lemma 2.9,  $i^2(r+1) + j^2r + 1 = 0$  has  $p - 1$  solutions. Hence  $(\mu_0, \mu_1, \mu_2) = (0, p^2, p+1)$ . If  $-(r+1)^{-1}r \notin (F_p^*)^2$ , then  $i^2(r+1) + r = 0$  has no solution, and by Lemma 2.9,  $i^2(r+1) + j^2r + 1 = 0$  has  $p + 1$  solutions. Hence we also have  $(\mu_0, \mu_1, \mu_2) = (0, p^2, p+1)$ .

If  $G$  is the group (D13), then  $p > 2$  and  $m_1 = m_2 = m_3 = 1$  and  $w(G) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ . By Theorem 4.5 (3),  $M \in \mathcal{A}_1$ ,  $M_i \in \mathcal{A}_2$  if and only if  $i^2 = 1$  and  $M_{ij} \in \mathcal{A}_2$  if and only if  $(i+1)^2 - j^2 - 1 = 0$ . By Lemma 2.9,  $(i+1)^2 - j^2 - 1 = 0$  has  $p - 1$  solutions. Hence  $(\mu_0, \mu_1, \mu_2) = (0, p^2, p+1)$ .

If  $G$  is the group (D14), then  $p > 2$  and  $m_1 = m_2 = m_3 = 1$  and  $w(G) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & 0 \end{pmatrix}$ . By Theorem 4.5 (3),  $M \in \mathcal{A}_2$ ,  $M_i \in \mathcal{A}_2$  and  $M_{ij} \in \mathcal{A}_1$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p^2, p+1)$ .

If  $G$  is the group (D15), then  $p > 2$  and  $m_1 = m_2 = m_3 = 1$  and  $w(G) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ . By Theorem 4.5 (3),  $M \in \mathcal{A}_2$ ,  $M_i \in \mathcal{A}_2$  and  $M_{ij} \in \mathcal{A}_1$  if and only if  $j = 0$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p^2 - p, 2p + 1)$ .

If  $G$  is the group (D16), then  $p > 2$  and  $m_1 = m_2 = m_3 = 1$  and  $w(G) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & r \end{pmatrix}$ . By Theorem 4.5 (3),  $M \in \mathcal{A}_1$ ,  $M_i \in \mathcal{A}_2$  if and only if  $i = 0$  and  $M_{ij} \in \mathcal{A}_1$  if and only if  $i = 0$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, p^2, p+1)$ .

If  $G$  is the group (D17), then  $p = 2$  and  $m_1 = m_2 = m_3 = 1$  and  $w(G) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ . By Theorem 4.5 (3),  $M \in \mathcal{A}_2$ ,  $M_i \in \mathcal{A}_2$  if and only if  $i = 0$  and  $M_{ij} \in \mathcal{A}_1$  if and only if  $j = 0$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, 3, 4)$ .

If  $G$  is the group (D18), then  $p = 2$  and  $m_1 = m_2 = m_3 = 1$  and  $w(G) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ . By Theorem 4.5 (3),  $M \in \mathcal{A}_2$ ,  $M_i \in \mathcal{A}_2$  if and only if  $i = 0$  and  $M_{ij} \in \mathcal{A}_1$  if and only if  $j = 0$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, 3, 4)$ .

If  $G$  is the group (D19), then  $p = 2$  and  $m_1 = m_2 = m_3 = 1$  and  $w(G) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ . By Theorem 4.5 (3),  $M \in \mathcal{A}_1$ ,  $M_i \in \mathcal{A}_2$  if and only if  $i = 0$  and  $M_{ij} \in \mathcal{A}_1$  if and only if  $i = j = 0$ . Hence  $(\mu_0, \mu_1, \mu_2) = (0, 5, 2)$ .  $\square$

**Theorem 4.7.**  *$G$  has no abelian subgroup of index  $p$  and  $G$  has a unique  $\mathcal{A}_1$ -subgroups of index  $p$  if and only if  $G$  is isomorphic to one of the following pairwise non-isomorphic groups:*

(Ei)  $p > 2$ . In this case  $p = 3$ ,  $(\mu_0, \mu_1, \mu_2) = (0, 1, 3)$  and  $\alpha_1(G) = 10$  except (E2), in which  $\alpha_1(G) = 28$ .

(E1)  $\langle s_1, s; s_2 \mid s_1^9 = s_2^9 = 1, s^3 = s_2^{3\delta}, [s_1, s] = s_2, [s_2, s] = s_2^{-3}s_1^{-3}, [s_2, s_1] = s_2^3 \rangle$  where  $\delta = 0, 1, 2$ ; moreover,  $|G| = 3^5$ ,  $c(G) = 4$ ,  $\Phi(G) = G' = \langle s_1^3, s_2 \rangle \cong C_3 \times C_9$  and  $Z(G) = \langle s_2^3 \rangle \cong C_3$ .

(E2)  $\langle a, b; c \mid a^{3^2} = b^{3^2} = c^3 = 1, [b, a] = c, [c, a] = a^3, [c, b] = b^{-3} \rangle$ ; where  $|G| = 3^5$ ,  $c(G) = 3$ ,  $\Phi(G) = G' = \langle a^3, b^3, c \rangle \cong C_3^3$  and  $Z(G) = \langle a^3, b^3 \rangle \cong C_3^2$ .

(E3)  $\langle s_1, \beta; s_2, x \mid s_1^9 = s_2^9 = x^3 = 1, \beta^3 = x, [s_1, \beta] = s_2, [s_2, \beta] = s_2^{-3}s_1^{-3}, [s_1, s_2] = x, [x, s_1] = [x, \beta] = 1 \rangle$ ; where  $|G| = 3^6$ ,  $c(G) = 4$ ,  $\Phi(G) = G' = \langle s_1^3, s_2, x \rangle \cong C_9 \times C_3 \times C_3$  and  $Z(G) = \langle s_2^3, x \rangle \cong C_3^2$ .

- (E4)  $\langle s_1, \beta; s_2, x \mid s_1^9 = s_2^9 = x^3 = 1, \beta^3 = s_2^3 x, [s_1, \beta] = s_2, [s_2, \beta] = s_2^{-3} s_1^{-3}, [s_1, s_2] = x, [x, s_1] = [x, \beta] = 1 \rangle$ ; where  $|G| = 3^6$ ,  $c(G) = 4$ ,  $\Phi(G) = G' = \langle s_1^3, s_2, x \rangle \cong C_9 \times C_3 \times C_3$  and  $Z(G) = \langle s_2^3, x \rangle \cong C_3^2$ .
- (E5)  $\langle \alpha, \beta; s_1, s_2, x \mid s_1^9 = s_2^3 = x^3 = 1, \beta^3 = x^2, \alpha^3 = s_2^{-1}, [\alpha, \beta] = s_1, [s_1, \alpha] = x, [s_1, \beta] = s_2, [s_2, \beta] = s_1^{-3}, [s_1, s_2] = [x, \alpha] = [x, \beta] = 1 \rangle$ ; where  $|G| = 3^6$ ,  $c(G) = 4$ ,  $\Phi(G) = G' = \langle s_1, s_2, x \rangle \cong C_9 \times C_3 \times C_3$  and  $Z(G) = \langle s_1^3, x \rangle \cong C_3^2$ .
- (E6)  $\langle \alpha, \beta; s_1, s_2, x \mid s_1^9 = s_2^3 = x^3 = 1, \beta^3 = x, \alpha^3 = s_2^{-1} x, [\alpha, \beta] = s_1, [s_1, \alpha] = x, [s_1, \beta] = s_2, [s_2, \beta] = s_1^{-3}, [s_1, s_2] = [x, \alpha] = [x, \beta] = 1 \rangle$ , where  $|G| = 3^6$ ,  $c(G) = 4$ ,  $\Phi(G) = G' = \langle s_1, s_2, x \rangle \cong C_9 \times C_3 \times C_3$  and  $Z(G) = \langle s_1^3, x \rangle \cong C_3^2$ .

(Eii)  $p = 2$ .

- (E7)  $\langle a, b \mid a^{16} = 1, b^{2^{s+t+2}} = a^{2^{s+t'+2}}, [a, b] = a^2 \rangle$ , where  $s, t, t'$  are non-negative integers with  $1 \leq s + t' \leq 2$ ,  $tt' = 0$  and if  $t' \neq 0$ , then  $s + t' = 2$ ; moreover,  $|G| = 2^{s+t+6}$ ,  $c(G) = 4$ ,  $\Phi(G) = \langle a^2, b^2 \rangle \cong C_8 \times C_{2^{t-t'+3}}$ ,  $G' = \langle a^2 \rangle \cong C_8$ ,  $Z(G) = \langle a^8, b^4 \rangle \cong C_2 \times C_{2^{t-t'+2}}$ ,  $(\mu_0, \mu_1, \mu_2) = (0, 1, 2)$  and  $\alpha_1(G) = 5$ ;
- (E8)  $\langle a, b, c \mid a^2 = b^8 = 1, c^2 = b^{4t}, [a, b] = b^4, [a, c] = 1, [c, b] = b^2 \rangle$  where  $t = 0, 1$ ; moreover,  $|G| = 2^5$ ,  $c(G) = 3$ ,  $\Phi(G) = G' = \langle b^2 \rangle \cong C_4$ ,  $Z(G) = \langle b^4 \rangle \cong C_2$ ,  $(\mu_0, \mu_1, \mu_2) = (0, 1, 6)$  and  $\alpha_1(G) = 9$ ;
- (E9)  $\langle a, b, c; d \mid a^4 = b^4 = c^2 = d^2 = 1, [a, b] = b^2, [c, a] = a^2 b^2, [c, b] = d, [d, a] = [d, b] = [d, c] = 1 \rangle$ ; moreover  $|G| = 2^6$ ,  $c(G) = 2$ ,  $\Phi(G) = Z(G) = G' = \langle a^2, b^2, d \rangle \cong C_2^3$ ,  $(\mu_0, \mu_1, \mu_2) = (0, 1, 6)$  and  $\alpha_1(G) = 25$ ;
- (E10)  $\langle a, b, c; d \mid a^4 = b^4 = c^2 = d^2 = 1, [a, b] = a^2, [c, a] = a^2 b^2, [c, b] = d, [d, a] = [d, b] = [d, c] = 1 \rangle$ , moreover  $|G| = 2^6$ ,  $c(G) = 2$ ,  $\Phi(G) = Z(G) = G' = \langle a^2, b^2, d \rangle \cong C_2^3$ ,  $(\mu_0, \mu_1, \mu_2) = (0, 1, 6)$  and  $\alpha_1(G) = 25$ .

**Proof** Finite  $p$ -groups with a unique  $\mathcal{A}_1$ -subgroup were classified in [22] for  $p > 2$  and [24] for  $p = 2$ . If  $p > 2$ , then, by using the classification in [22] and calculating the maximal index of  $\mathcal{A}_1$ -subgroups for these groups, we get the groups (E1)–(E6). The details are omitted. Table 8 gives the correspondence.

Groups	Groups in [22, Lemma 2.12]	Groups	Groups in [22, Corollary 3.5]	Groups	Groups in [22, Theorem 3.8]
(E1)	(2) where $e = 2$	(E2)	(2)	(E3)	(1) where $e = 2$ and $k = 1$
				(E4)	(2) where $e = 2$ and $k = 1$
				(E5)	(3) where $e = 2$ and $k = 2$
				(E6)	(4) where $e = 2$ and $k = 1$

Table 8: The correspondence from Theorem 4.6 to [22, Lemma 2.12, Corollary 3.5, Theorem 3.8]

If  $p = 2$ , then, by lemma 3.7,  $G$  is either metacyclic or  $|G| \leq 2^8$ . If  $G$  is metacyclic, then, by Lemma 3.1,  $|G'| = p^3$ . By using the classification of metacyclic 2-groups in [30] and checking the number of their  $\mathcal{A}_1$ -subgroups of index  $p$ , we get the group (E7). The details is omitted. If  $G$  is not metacyclic, then  $|G| \leq 2^8$ . By using Magma to check the SmallGroup database, we get the groups (E8)–(E10).



We calculate the  $(\mu_0, \mu_1, \mu_2)$  and  $\alpha_1(G)$  of those groups in Theorem 4.7 as follows.

By hypothesis,  $\mu_0 = 0$  and  $\mu_1 = 1$ . Hence  $\mu_2 = p$  for  $d(G) = 2$  and  $\mu_2 = p + p^2$  for  $d(G) = 3$ . In the following, we calculate  $\alpha_1(G)$ .

If  $G$  is the group (E2), then  $|G| = 3^5$ , and  $|H'| = p$  for any maximal subgroup  $H$ . By Lemma 2.6 (7),  $\alpha_1(H) = p^2$  for  $H \in \mathcal{A}_2$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = \mu_1 + p^2 \mu_2 = p^3 + 1 = 28.$$

If  $G$  is one of the group (E1) and (E3)–(E6), then  $c(G) = 4$ , and  $c(H) = 3$  for any  $\mathcal{A}_2$ -subgroup  $H$ . Since  $d(H) = 2$ ,  $\alpha_1(H) = p$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = \mu_1 + p \mu_2 = p^2 + 1 = 10.$$

If  $G$  is the group (E7), then  $G$  is metacyclic. Hence  $H$  is metacyclic for any  $\mathcal{A}_2$ -subgroup  $H$ . Since  $d(H) = 2$ ,  $\alpha_1(H) = p$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = \mu_1 + p \mu_2 = p^2 + 1 = 5.$$

If  $G$  is the group (E8), then  $d(G) = 3$  and  $c(G) = 3$ . All maximal subgroups of  $G$  are  $M = \langle b, a \rangle$ ,  $M_0 = \langle c, a, b^2 \rangle$ ,  $M_1 = \langle cb, a, b^2 \rangle$  and  $M_{ij} = \langle ca^i, ba^j \rangle$  where  $i, j = 0, 1$ . Here  $M$  is the unique  $\mathcal{A}_1$ -subgroup of index 2,  $|M'_i| = 2$  and  $c(M_{ij}) = 3$ . By Lemma 2.6,  $\alpha_1(M_i) = 4$  and  $\alpha_1(M_{ij}) = 2$ . Let  $H \in \Gamma_2(G)$ . Then  $H = \langle a^i b^j c^k, b^2 \rangle$  where  $(i, j, k) \neq (0, 0, 0)$ . It is obvious that  $H \in \mathcal{A}_1$  if and only if  $k = 1$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) - 2 \sum_{H \in \Gamma_2} \alpha_1(H) = 1 + 4 \times 2 + 2 \times 4 - 2 \times 4 = 9.$$

If  $G$  is the group (E9) or (E10), then  $d(G) = 3$  and  $\Phi(G) = Z(G)$ . By Theorem 3.18,  $\alpha_1(G) = \mu_1 + 4\mu_2 = 25$ .  $\square$

## 5 $\mathcal{A}_3$ -groups without $\mathcal{A}_1$ -subgroup of index $p$

Assume  $G$  is an  $\mathcal{A}_3$ -group without an  $\mathcal{A}_1$ -subgroup of index  $p$ . We discuss according to  $G$  has an abelian subgroup of index  $p$  or not. In this case,  $\mathcal{A}_3$ -groups have 151 non-isomorphic types. Theorem 5.1, 5.2, 5.3, 5.4 and 5.5 give the classification of  $\mathcal{A}_3$ -groups without an  $\mathcal{A}_1$ -subgroup of index  $p$  and with an abelian subgroup of index  $p$ . They are the groups (F1)–(F8), (G1)–(G12), (H1)–(H10), (I1)–(I11) and (J1)–(J9). Theorem 5.6, 5.7, 5.13, 5.15 and 5.16 give the classification of  $\mathcal{A}_3$ -groups without an  $\mathcal{A}_1$ -subgroup of index  $p$  and an abelian subgroup of index  $p$ . They are the groups (K1)–(K5), (L1)–(L2), (M1)–(M62), (N1)–(N26) and (O1)–(O6).

### 5.1 $G$ has an abelian subgroup of index $p$

In this section assume  $G$  is an  $\mathcal{A}_3$ -group with an abelian subgroup of index  $p$ , and  $G$  without an  $\mathcal{A}_1$ -subgroup of index  $p$  in Theorem 5.1, 5.2, 5.3, 5.4 and 5.5.

**Theorem 5.1.**  $d(G) = 2$  and  $c(G) = 4$  if and only if  $G$  is isomorphic to one of the following pairwise non-isomorphic groups:

- (F1)  $\langle a, b \mid a^{16} = b^{2^m} = 1, [a, b] = a^{-2} \rangle$ ; where  $|G| = 2^{m+4}$ ,  $\Phi(G) = \langle a^2, b^2 \rangle \cong C_8 \times C_{2^{m-1}}$  if  $m > 1$ ,  $\Phi(G) \cong C_8$  if  $m = 1$ ,  $G' = \langle a^2 \rangle \cong C_8$ ,  $Z(G) = \langle a^8, b^2 \rangle \cong C_2 \times C_{2^{m-1}}$  if  $m > 1$ ,  $Z(G) \cong C_2$  if  $m = 1$ .
- (F2)  $\langle a, b \mid a^{16} = b^{2^m} = 1, [a, b] = a^6 \rangle$ ; where  $|G| = 2^{m+4}$ ,  $\Phi(G) = \langle a^2, b^2 \rangle \cong C_8 \times C_{2^{m-1}}$  if  $m > 1$ ,  $\Phi(G) \cong C_8$  if  $m = 1$ ,  $G' = \langle a^2 \rangle \cong C_8$ ,  $Z(G) = \langle a^8, b^2 \rangle \cong C_2 \times C_{2^{m-1}}$  if  $m > 1$ ,  $Z(G) \cong C_2$  if  $m = 1$ .
- (F3)  $\langle a, b \mid a^{16} = 1, b^{2^m} = a^8, [a, b] = a^{-2} \rangle$ ; where  $|G| = 2^{m+4}$ ,  $\Phi(G) = \langle a^2, b^2 \rangle \cong C_2^3$  if  $m = 1$ ,  $\Phi(G) \cong C_2^3 \times C_2$  if  $m = 2$ ,  $\Phi(G) \cong C_{2^m} \times C_{2^2}$  if  $m > 2$ ,  $G' = \langle a^2 \rangle \cong C_8$  and  $Z(G) = \langle b^2 \rangle \cong C_{2^m}$ .
- (F4)  $\langle a_1, b; a_2 \mid a_1^9 = a_2^9 = b^{3^m} = 1, [a_1, b] = a_2, [a_2, b] = a_1^{-3} a_2^{3t}, [a_1, a_2] = 1 \rangle$ , where  $t = 1, 2$ ; moreover,  $|G| = 3^{m+4}$ ,  $\Phi(G) = \langle a_2, a_1^3, b^3 \rangle \cong C_3 \times C_9 \times C_{3^{m-1}}$  if  $m > 1$ ,  $\Phi(G) \cong C_3 \times C_9$  if  $m = 1$ ,  $G' = \langle a_2, a_1^3 \rangle \cong C_3 \times C_9$ ,  $Z(G) = \langle a_2^3, b^3 \rangle \cong C_3 \times C_{3^{m-1}}$  if  $m > 1$ ,  $Z(G) \cong C_3$  if  $m = 1$ .
- (F5)  $\langle a_1, b; a_2 \mid a_1^9 = a_2^9 = 1, b^{3^m} = a_2^{-3}, [a_1, b] = a_2, [a_2, b] = a_1^{-3} a_2^{-3}, [a_1, a_2] = 1 \rangle$ ; where  $|G| = 3^{m+4}$ ,  $\Phi(G) = \langle a_2, a_1^3, b^3 \rangle \cong C_9 \times C_3$  if  $m = 1$ ,  $\Phi(G) \cong C_{3^m} \times C_3 \times C_3$  if  $m > 1$ ,  $G' = \langle a_2, a_1^3 \rangle \cong C_9 \times C_3$  and  $Z(G) = \langle b^3 \rangle \cong C_{3^m}$ .
- (F6)  $\langle a_1, b; a_2, a_3, a_4 \mid a_i^p = b^{p^m} = 1, [a_j, b] = a_{j+1}, [a_4, b] = 1, [a_i, a_j] = 1 \rangle$ , where  $p \geq 5$ ,  $1 \leq i \leq 4$ ,  $1 \leq j \leq 3$ ; moreover,  $|G| = p^{m+4}$ ,  $\Phi(G) = \langle a_2, a_3, a_4, b^p \rangle \cong C_p^3 \times C_{p^{m-1}}$  if  $m > 1$ ,  $\Phi(G) \cong C_p^3$  if  $m = 1$ ,  $G' = \langle a_2, a_3, a_4 \rangle \cong C_p^3$ ,  $Z(G) = \langle a_4, b^p \rangle \cong C_p \times C_{p^{m-1}}$  if  $m > 1$ ,  $Z(G) \cong C_p$  if  $m = 1$ .
- (F7)  $\langle a_1, b; a_2, a_3 \mid a_i^p = b^{p^{m+1}} = 1, [a_j, b] = a_{j+1}, [a_3, b] = b^{p^m}, [a_i, a_j] = 1 \rangle$ , where  $p \geq 5$ ,  $1 \leq i \leq 3$ ,  $1 \leq j \leq 2$ ; moreover,  $|G| = p^{m+4}$ ,  $\Phi(G) = \langle a_2, a_3, b^p \rangle \cong C_p^2 \times C_{p^m}$ ,  $G' = \langle a_2, a_3, b^{p^m} \rangle \cong C_p^3$  and  $Z(G) = \langle b^p \rangle \cong C_{p^m}$ .
- (F8)  $\langle a_1, b; a_2, a_3 \mid a_1^{p^2} = a_i^p = b^{p^m} = 1, [a_j, b] = a_{j+1}, [a_3, b] = a_1^{tp}, [a_i, a_j] = 1 \rangle$ , where  $2 \leq i \leq 3$ ,  $1 \leq j \leq 2$ , and  $t = t_1, t_2, \dots, t_{(3,p-1)}$ , where  $p \geq 5$ ,  $t_1, t_2, \dots, t_{(3,p-1)}$  are the coset representatives of the subgroup  $(F_p^*)^3$  in  $F_p^*$ . Moreover,  $|G| = p^{m+4}$ ,  $\Phi(G) = \langle a_2, a_3, a_1^p, b^p \rangle \cong C_p^3 \times C_{p^{m-1}}$  if  $m > 1$ ,  $\Phi(G) \cong C_p^3$  if  $m = 1$ ,  $G' = \langle a_2, a_3, a_1^p \rangle \cong C_p^3$ ,  $Z(G) = \langle a_1^p, b^p \rangle \cong C_p \times C_{p^{m-1}}$  if  $m > 1$ ,  $Z(G) \cong C_p$  if  $m = 1$ .

Moreover,  $(\mu_0, \mu_1, \mu_2) = (1, 0, p)$  and  $\alpha_1(G) = p^2$ .

**Proof** Let  $A$  be an abelian subgroup of index  $p$  and  $B$  be a non-abelian subgroup of index  $p$ . By Lemma 2.11,  $G_3 = B'$ ,  $G_4 = B_3$ ,  $|G_3/G_4| = p$  and  $|G'| = p^{c(G)-1} = p^3$ . Since  $c(G) = 4$ ,  $c(B) \geq 3$ . Hence Lemma 2.6 gives that  $d(B) = 2$  and  $c(B) = 3$ . By arbitrariness of  $B$ , all non-abelian subgroups of  $G$  are generated by two elements. Such groups were classified by [29]. It follows from [29, Main Theorem] that  $G$  is either a  $p$ -groups of maximal class or one of the groups listed in [29, Theorem 3.12-3.13]. Since  $c(G) = 4$ , they are the groups (F1)–(F8).

Since  $d(G) = 2$ ,  $G$  has  $1+p$  maximal subgroups. Hence  $(\mu_0, \mu_1, \mu_2) = (1, 0, p)$ . Since  $d(H) = 2$  for any non-abelian maximal subgroup  $H$ ,  $\alpha_1(H) = p$ . By Hall's enumeration principle,  $\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = p\mu_2 = p^2$ .  $\square$

**Theorem 5.2.**  $d(G) = 2$  and  $c(G) = 3$  if and only if  $G$  is isomorphic to one of the following pairwise non-isomorphic groups:

(Gi)  $G' \cong C_{p^2}$ .

(G1)  $\langle a, b, c \mid a^8 = 1, c^2 = a^4 = b^4, [a, b] = c, [c, a] = 1, [c, b] = c^2 \rangle$ ; where  $|G| = 2^6$ ,  $\Phi(G) = \langle a^2, b^2, c \rangle \cong C_2 \times C_2 \times C_4$ ,  $G' = \langle c \rangle$ ,  $Z(G) = \langle ca^2, b^2 \rangle \cong C_2 \times C_4$ .

(G2)  $\langle a, b, c \mid a^8 = b^4 = 1, c^2 = a^4, [a, b] = c, [c, a] = 1, [c, b] = c^2 \rangle$ ; where  $|G| = 2^6$ ,  $\Phi(G) = \langle a^2, b^2, c \rangle \cong C_2 \times C_2 \times C_4$ ,  $G' = \langle c \rangle$ ,  $Z(G) = \langle ca^2, b^2 \rangle \cong C_2^2$ .

(G3)  $\langle a, b, c \mid a^8 = b^4 = 1, c^2 = a^4, [a, b] = c, [c, a] = c^2, [c, b] = 1 \rangle$ ; where  $|G| = 2^6$ ,  $\Phi(G) = \langle a^2, b^2, c \rangle \cong C_4 \times C_2^2$ ,  $G' = \langle c \rangle$  and  $Z(G) = \langle a^2, cb^2 \rangle \cong C_4 \times C_2$ .

(G4)  $\langle a, b, c \mid a^{2^{n+1}} = b^4 = 1, c^2 = a^{2^n}, [a, b] = c, [c, a] = c^2, [c, b] = 1 \rangle$ , where  $n > 2$ ; moreover,  $|G| = 2^{n+4}$ ,  $\Phi(G) = \langle a^2, b^2, c \rangle \cong C_{2^n} \times C_2^2$ ,  $G' = \langle c \rangle$  and  $Z(G) = \langle a^2, cb^2 \rangle \cong C_{2^n} \times C_2$ .

(G5)  $\langle a, b, c \mid a^{2^n} = b^8 = 1, c^2 = b^4, [a, b] = c, [c, a] = c^2, [c, b] = 1 \rangle$ , where  $n > 2$ ; moreover,  $|G| = 2^{n+4}$ ,  $\Phi(G) = \langle a^2, b^2, c \rangle \cong C_{2^{n-1}} \times C_2 \times C_4$ ,  $G' = \langle c \rangle$  and  $Z(G) = \langle a^2, cb^2 \rangle \cong C_{2^{n-1}} \times C_2$ .

(G6)  $\langle a, b, c \mid a^{2^n} = b^4 = c^4 = 1, [a, b] = c, [c, a] = c^2, [c, b] = 1 \rangle$ , where  $n > 2$ . Moreover,  $|G| = 2^{n+4}$ ,  $\Phi(G) = \langle a^2, b^2, c \rangle \cong C_{2^{n-1}} \times C_2 \times C_4$ ,  $G' = \langle c \rangle$  and  $Z(G) = \langle a^2, cb^2 \rangle \cong C_{2^{n-1}} \times C_4$ .

(Gii)  $G' \cong C_p^2$ .

(G7)  $\langle a, b, c \mid a^{p^3} = b^{p^2} = c^p = 1, [a, b] = c, [c, a] = 1, [c, b] = a^{\nu p^2} \rangle$ , where  $p > 2$  and  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ ; moreover,  $|G| = p^6$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^2} \times C_p^2$ ,  $G' = \langle a^{p^2}, c \rangle$  and  $Z(G) = \langle a^p, b^p \rangle \cong C_{p^2} \times C_p$ .

(G8)  $\langle a, b, c, d \mid a^{p^2} = b^{p^2} = c^p = d^p = 1, [a, b] = c, [c, a] = d, [c, b] = 1, [d, a] = [d, b] = 1 \rangle$ , where  $p > 2$ . Moreover,  $|G| = p^6$ ,  $\Phi(G) = \langle a^p, b^p, c, d \rangle \cong C_p^4$ ,  $G' = \langle c, d \rangle$  and  $Z(G) = \langle a^p, b^p, d \rangle \cong C_p^3$ .

- (G9)  $\langle a, b, c \mid a^{p^3} = b^{p^2} = c^p = 1, [a, b] = c, [c, a] = a^{p^2}, [c, b] = 1 \rangle$ , where  $p > 2$ ; moreover,  $|G| = p^6$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^2} \times C_p^2$ ,  $G' = \langle a^{p^2}, c \rangle$  and  $Z(G) = \langle a^p, b^p \rangle \cong C_{p^2} \times C_p$ .
- (G10)  $\langle a, b, c \mid a^{p^{n+1}} = b^{p^2} = c^p = 1, [a, b] = c, [c, a] = a^{p^n}, [c, b] = 1 \rangle$ , where  $p > 2$  and  $n > 2$ ; moreover,  $|G| = p^{n+4}$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^n} \times C_p^2$ ,  $G' = \langle a^{p^n}, c \rangle$  and  $Z(G) = \langle a^p, b^p \rangle \cong C_{p^n} \times C_p$ .
- (G11)  $\langle a, b, c \mid a^{p^n} = b^{p^3} = c^p = 1, [a, b] = c, [c, a] = b^{\nu p^2}, [c, b] = 1 \rangle$ , where  $p > 2$ ,  $n > 2$ , and  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ ; moreover,  $|G| = p^{n+4}$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^{n-1}} \times C_{p^2} \times C_p$ ,  $G' = \langle b^{p^2}, c \rangle$  and  $Z(G) = \langle a^p, b^p \rangle \cong C_{p^{n-1}} \times C_{p^2}$ .
- (G12)  $\langle a, b, c, d \mid a^{p^n} = b^{p^2} = c^p = d^p = 1, [a, b] = c, [c, a] = d, [c, b] = 1, [d, a] = [d, b] = 1 \rangle$ , where  $p > 2$  and  $n > 2$ . Moreover,  $|G| = p^{n+4}$ ,  $\Phi(G) = \langle a^p, b^p, c, d \rangle \cong C_{p^{n-1}} \times C_p^3$ ,  $G' = \langle c, d \rangle$  and  $Z(G) = \langle a^p, b^p, d \rangle \cong C_{p^{n-1}} \times C_p^2$ .

Moreover,  $(\mu_0, \mu_1, \mu_2) = (1, 0, p)$  and  $\alpha_1(G) = p^3$ .

**Proof** Let  $A$  be an abelian subgroup of index  $p$  and  $B$  be a non-abelian subgroup of index  $p$ . By Lemma 2.11,  $G_3 = B'$ ,  $G_4 = B_3$ ,  $|G_3/G_4| = p$  and  $|G'| = p^{c(G)-1}$ .

Since  $c(G) = 3$ ,  $|B'| = |G_3| = p$  and  $|G'| = p^2$ . Let  $\bar{G} = G/G_3$ . Then  $|\bar{G}'| = p$  and  $d(\bar{G}) = 2$ . Hence  $\bar{G}$  is an  $\mathcal{A}_1$ -group by Lemma 2.2. If  $\bar{G}$  is metacyclic, then  $G$  is also metacyclic by Lemma 2.1 and hence  $G$  is an  $\mathcal{A}_2$ -group by Lemma 3.1. This contradicts that  $G$  is an  $\mathcal{A}_3$ -group. Thus  $\bar{G}$  is a non-metacyclic  $\mathcal{A}_1$ -group. If  $G' \cong C_{p^2}$ , then  $G$  is one of the groups listed in [3, Theorem 3.5]. By [3, Theorem 3.1 & 3.6] we get groups (G1)–(G6). If  $G' \cong C_p^2$ , then  $G$  is one of the groups listed in [3, Theorem 4.6]. By [3, Theorem 4.1 & 4.7] we get groups (G7)–(G12).

Groups	Groups in [3, Theorem 3.5]	Groups	Groups in [3, Theorem 4.6]
(G1)	(C3)	(G7)	(G1) where $m = 2$ and $p > 2$
(G2)	(C4)	(G8)	(G2) where $m = 2$ and $p > 2$
(G3)	(C5)	(G9)	(G3) where $m = 2$ and $p > 2$
(G4)	(E2) where $p = m = 2$	(G10)	(J2) where $m = 2$ and $p > 2$
(G5)	(E5) where $p = m = 2$	(G11)	(J4) where $m = 2$ and $p > 2$
(G6)	(E8) where $p = m = 2$	(G12)	(J6) where $m = 2$ and $p > 2$

Table 9: The correspondence from Theorem 5.2 to [3, Theorem 3.5 & 4.6]

Now we calculate the  $(\mu_0, \mu_1, \mu_2)$  and  $\alpha_1(G)$ . Since  $d(G) = 2$ ,  $G$  has  $1 + p$  maximal subgroups. Hence  $(\mu_0, \mu_1, \mu_2) = (1, 0, p)$ . Since  $|H'| = p$  for any non-abelian maximal subgroup  $H$ , by Lemma 2.6 (7),  $\alpha_1(H) = p^2$ . By Hall's enumeration principle,  $\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = p^2 \mu_2 = p^3$ .  $\square$

**Theorem 5.3.**  $d(G) = 3$  and  $\Phi(G) \leq Z(G)$  if and only if  $G$  is isomorphic to one of the following pairwise non-isomorphic groups:

(Hi)  $\Phi(G) < Z(G)$ ,  $G' \cong C_p$  and  $c(G) = 2$ . Moreover,  $(\mu_0, \mu_1, \mu_2) = (1 + p, 0, p^2)$  and  $\alpha_1(G) = p^4$ .

(H1)  $\langle a, b, c \mid a^{p^{n+1}} = b^{p^m} = c^{p^2} = 1, [a, b] = a^{p^n}, [c, a] = [c, b] = 1 \rangle \cong M_p(n + 1, m) \times C_{p^2}$ , where  $\min\{n, m\} \geq 2$ ; moreover,  $|G| = p^{n+m+3}$ ,  $\Phi(G) = \langle a^p, b^p, c^p \rangle \cong C_{p^n} \times C_{p^{m-1}} \times C_p$ ,  $G' = \langle a^{p^n} \rangle$  and  $Z(G) = \langle a^p, b^p, c \rangle \cong C_{p^n} \times C_{p^{m-1}} \times C_{p^2}$ .

(H2)  $\langle a, b, c, d \mid a^{p^n} = b^{p^m} = c^{p^2} = d^p = 1, [a, b] = d, [c, a] = [c, b] = 1 \rangle \cong M_p(n, m, 1) \times C_{p^2}$ , where  $n \geq m \geq 2$ ; moreover,  $|G| = p^{n+m+3}$ ,  $\Phi(G) = \langle a^p, b^p, c^p, d \rangle \cong C_{p^{n-1}} \times C_{p^{m-1}} \times C_p \times C_p$ ,  $G' = \langle d \rangle$  and  $Z(G) = \langle a^p, b^p, c, d \rangle \cong C_{p^{n-1}} \times C_{p^{m-1}} \times C_{p^2} \times C_p$ .

(H3)  $\langle a, b, c \mid a^{p^n} = b^{p^m} = c^{p^3} = 1, [a, b] = c^{p^2}, [c, a] = [c, b] = 1 \rangle \cong M_p(n, m, 1) * C_{p^3}$ , where  $n \geq m \geq 2$ . Moreover,  $|G| = p^{n+m+3}$ ,  $\Phi(G) = \langle a^p, b^p, c^p \rangle \cong C_{p^{n-1}} \times C_{p^{m-1}} \times C_{p^2}$ ,  $G' = \langle c^{p^2} \rangle$  and  $Z(G) = \langle a^p, b^p, c \rangle \cong C_{p^{n-1}} \times C_{p^{m-1}} \times C_{p^3}$ .

(Hii)  $\Phi(G) = Z(G)$ ,  $G' \cong C_p^2$  and  $c(G) = 2$ . Moreover,  $(\mu_0, \mu_1, \mu_2) = (1, 0, p + p^2)$  and  $\alpha_1(G) = p^4 + p^3$ .

(H4)  $\langle a, b, c \mid a^{p^l} = b^{p^2} = c^{p^2} = 1, [b, c] = 1, [c, a] = c^p, [a, b] = b^{-p} \rangle$ , where  $p > 2$  and  $l \geq 2$ ; moreover,  $|G| = p^{l+4}$ ,  $\Phi(G) = Z(G) = \langle a^p, b^p, c^p \rangle \cong C_{p^{l-1}} \times C_p^2$ ,  $G' = \langle b^p, c^p \rangle$ .

(H5)  $\langle a, b, c \mid a^{p^l} = b^{p^3} = c^{p^3} = 1, [b, c] = 1, [c, a] = b^{p^2} c^{tp^2}, [a, b] = b^{-tp^2} c^{\nu p^2} \rangle$ , where  $p > 2$ ,  $l \geq 2$ ,  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$  such that  $-\nu \notin (F_p^*)^2$  and  $t \in \{0, 1, \dots, \frac{p-1}{2}\}$ ; moreover,  $|G| = p^{l+6}$ ,  $\Phi(G) = Z(G) = \langle a^p, b^p, c^p \rangle \cong C_{p^{l-1}} \times C_{p^2} \times C_{p^2}$ ,  $G' = \langle b^{p^2}, c^{p^2} \rangle$ .

(H6)  $\langle a, b, c \mid a^{2^l} = b^4 = c^4 = 1, [b, c] = 1, [c, a] = c^2, [a, b] = b^2 \rangle$ ; moreover,  $|G| = 2^{l+4}$ ,  $\Phi(G) = Z(G) = \langle a^2, b^2, c^2 \rangle \cong C_{2^{l-1}} \times C_2 \times C_2$  if  $l > 1$ ,  $G' = \langle b^2, c^2 \rangle$ ,  $\Phi(G) = Z(G) \cong C_2 \times C_2$  if  $l = 1$ .

(H7)  $\langle a, b, c \mid a^{2^l} = b^8 = c^8 = 1, [b, c] = 1, [c, a] = b^4, [a, b] = b^4 c^4 \rangle$ , where  $l \geq 2$ ; moreover,  $|G| = 2^{l+6}$ ,  $\Phi(G) = Z(G) = \langle a^2, b^2, c^2 \rangle \cong C_{2^{l-1}} \times C_4 \times C_4$ ,  $G' = \langle b^4, c^4 \rangle$ .

(H8)  $\langle a, b, c \mid a^{p^{l+1}} = b^{p^3} = c^{p^2} = 1, [b, c] = 1, [c, a] = b^{p^2}, [a, b] = a^{p^l} \rangle$ , where  $l \geq 2$ ; moreover,  $|G| = p^{l+6}$ ,  $\Phi(G) = Z(G) = \langle a^p, b^p, c^p \rangle \cong C_{p^l} \times C_{p^2} \times C_p$ ,  $G' = \langle a^{p^l}, b^{p^2} \rangle$ .

(H9)  $\langle a, b, c, x \mid a^{p^l} = b^{p^2} = c^{p^2} = x^p = 1, [a, b] = c^p, [a, c] = x, [b, c] = [x, a] = [x, b] = [x, c] = 1 \rangle$ , where  $l \geq 2$ ; moreover,  $|G| = p^{l+5}$ ,  $\Phi(G) = Z(G) = \langle a^p, b^p, c^p, x \rangle \cong C_{p^{l-1}} \times C_p^3$ ,  $G' = \langle c^p, x \rangle$ .

(H10)  $\langle a, b, c, x, y \mid a^{p^l} = b^p = c^p = x^p = y^p = 1, [a, b] = x, [a, c] = y, [b, c] = [x, a] = [x, b] = [x, c] = [y, a] = [y, b] = [y, c] = 1 \rangle$ , where  $l \geq 2$  if  $p = 2$ ; moreover,  $|G| = p^{l+4}$ ,  $\Phi(G) = Z(G) = \langle a^p, x, y \rangle \cong C_{p^{l-1}} \times C_p^2$ ,  $G' = \langle x, y \rangle$ .

**Proof** By Lemma 2.8,  $|G'| \leq p^2$ . If  $|G'| = p$ , then  $G$  is one of the groups listed in [2, Theorem 3.1]. Since  $G$  has no  $\mathcal{A}_1$ -subgroup of index  $p$ , they are the groups (H1)–(H3). If  $|G'| = p^2$ , then  $G$  is one of the groups listed in [2, Theorem 4.8]. By checking [2, Table 4] we get groups (H4)–(H10).

Groups	Groups in [2, Theorem 4.8]	Groups	Groups in [2, Theorem 4.8]
(H4)	(A1) where $l \geq 2$ and $m = 1$	(H8)	(A10) where $l \geq 2 = m = n$
(H5)	(A3) where $l \geq m = 2$ and $-\nu \notin F_p^2$	(H9)	(B2) where $m = 2$ and $n = 1$
(H6)	(A5) where $m = 1$	(H10)	(C) where $m = n = 1$
(H7)	(A6) where $l \geq m = 2$ and $-\nu \notin (F_p^*)^2$		

Table 10: The correspondence from Theorem 5.3 to [2, Theorem 4.8]

Now we calculate the  $(\mu_0, \mu_1, \mu_2)$  and  $\alpha_1(G)$ . If  $G$  is one of the groups (H1)–(H3), then  $\mu_0 = 1 + p$ . Since  $\mu_1 = 0$ ,  $\mu_2(G) = p^2$ . By Lemma 3.18,  $\alpha_1(G) = p^2\mu_2 = p^4$ .

If  $G$  is one of the groups (H4)–(H10), then  $d(G) = 3$  and  $G' \cong C_p^2$ . Since  $\Phi(G) = Z(G)$ ,  $\mu_0 = 1$ . Hence  $(\mu_0, \mu_1, \mu_2) = (1, 0, p + p^2)$ . By Lemma 3.18,  $\alpha_1(G) = p^2\mu_2 = p^4 + p^3$ .  $\square$

**Theorem 5.4.**  $d(G) = 3$  and  $\Phi(G) \not\leq Z(G)$  if and only if  $G$  is isomorphic to one of the following pairwise non-isomorphic groups:

- (Ii)  $|G'| = p^2$  and  $c(G) = 3$ . In this case,  $Z(G) \not\leq \Phi(G)$ ,  $(\mu_0, \mu_1, \mu_2) = (1, 0, p^2 + p)$  and  $\alpha_1(G) = p^3$ .
- (II)  $\langle a, b, x \mid a^8 = b^{2^m} = x^2 = 1, [a, b] = a^{-2}, [x, a] = [x, b] = 1 \rangle = \langle a, b \rangle \times \langle x \rangle$ ;  
where  $|G| = 2^{m+4}$ ,  $\Phi(G) = \langle a^2, b^2 \rangle \cong C_{2^{m-1}} \times C_4$  if  $m > 1$ ,  $\Phi(G) \cong C_4$  if  $m = 1$ ,  $G' = \langle a^2 \rangle \cong C_4$ ,  $Z(G) = \langle a^4, b^2, x \rangle \cong C_{2^{m-1}} \times C_2^2$  if  $m > 1$ ,  $Z(G) \cong C_2^2$  if  $m = 1$ .
- (I2)  $\langle a, b, x \mid a^8 = b^{2^m} = 1, x^2 = a^4, [a, b] = a^{-2}, [x, a] = [x, b] = 1 \rangle = \langle a, b \rangle * \langle x \rangle$ ;  
where  $|G| = 2^{m+4}$ ,  $\Phi(G) = \langle a^2, b^2 \rangle \cong C_{2^{m-1}} \times C_4$  if  $m > 1$ ,  $\Phi(G) \cong C_4$  if  $m = 1$ ,  $G' = \langle a^2 \rangle \cong C_4$ ,  $Z(G) = \langle b^2, x \rangle \cong C_{2^{m-1}} \times C_4$  if  $m > 1$ ,  $Z(G) \cong C_4$  if  $m = 1$ .
- (I3)  $\langle a, b, x \mid a^8 = b^{2^m} = x^2 = 1, [a, b] = a^2, [x, a] = [x, b] = 1 \rangle = \langle a, b \rangle \times \langle x \rangle$ ;  
where  $|G| = 2^{m+4}$ ,  $\Phi(G) = \langle a^2, b^2 \rangle \cong C_{2^{m-1}} \times C_4$  if  $m > 1$ ,  $\Phi(G) \cong C_4$  if  $m = 1$ ,  $G' = \langle a^2 \rangle \cong C_4$ ,  $Z(G) = \langle a^4, b^2, x \rangle \cong C_{2^{m-1}} \times C_2^2$  if  $m > 1$ ,  $Z(G) \cong C_2^2$  if  $m = 1$ .
- (I4)  $\langle a, b, x \mid a^8 = x^2 = 1, b^{2^m} = a^4, [a, b] = a^{-2}, [x, a] = [x, b] = 1 \rangle = \langle a, b \rangle \times \langle x \rangle$ ;  
where  $|G| = 2^{m+4}$ ,  $\Phi(G) = \langle a^2, b^2 \rangle \cong C_4$  if  $m = 1$ ,  $\Phi(G) = \langle a^2, b^2 \rangle \cong C_{2^m} \times C_2$  if  $m > 1$ ,  $G' = \langle a^2 \rangle \cong C_4$  and  $Z(G) = \langle b^2, x \rangle \cong C_{2^m} \times C_2$ .
- (I5)  $\langle a_1, b, x; a_2, a_3 \mid a_1^p = a_2^p = a_3^p = b^{p^m} = x^p = 1, [a_1, b] = a_2, [a_2, b] = a_3, [a_2, a_1] = [a_3, a_1] = [a_3, a_2] = [x, a_1] = [x, b] = 1 \rangle = \langle a_1, b \rangle \times \langle x \rangle$ ,  
where  $p > 3$  for  $m = 1$  and  $p > 2$ ; moreover,  $|G| = p^{m+4}$ ,  $\Phi(G) =$

- $\langle a_2, a_3, b^p \rangle \cong C_{p^{m-1}} \times C_p^2$  if  $m > 1$ ,  $\Phi(G) \cong C_p^2$  if  $m = 1$ ,  $G' = \langle a_2, a_3 \rangle \cong C_p^2$ ,  $Z(G) = \langle a_3, b^p, x \rangle \cong C_{p^{m-1}} \times C_p^2$  if  $m > 1$ ,  $Z(G) \cong C_p^2$  if  $m = 1$ .
- (I6)  $\langle a_1, x, b; a_2 \mid a_1^p = a_2^p = b^{p^2} = x^{p^2} = 1, [a_1, b] = a_2, [a_2, b] = x^p, [a_2, a_1] = [x, a_1] = [x, b] = 1 \rangle = \langle a_1, b \rangle * \langle x \rangle$ , where  $p > 2$ ; moreover,  $|G| = p^{m+4}$ ,  $\Phi(G) = \langle a_2, b^p, x^p \rangle \cong C_{p^{m-1}} \times C_p^2$  if  $m > 1$ ,  $\Phi(G) \cong C_p^2$  if  $m = 1$ ,  $G' = \langle a_2, x^p \rangle \cong C_p^2$ ,  $Z(G) = \langle x, b^p \rangle \cong C_{p^{m-1}} \times C_{p^2}$  if  $m > 1$ ,  $Z(G) \cong C_{p^2}$  if  $m = 1$ .
- (I7)  $\langle a_1, x, b; a_2 \mid a_1^p = a_2^p = b^{p^{m+1}} = x^p = 1, [a_1, b] = a_2, [a_2, b] = b^{p^m}, [a_2, a_1] = [x, a_1] = [x, b] = 1 \rangle = \langle a_1, b \rangle \times \langle x \rangle$ , where  $p > 2$ ; moreover,  $|G| = p^{m+4}$ ,  $\Phi(G) = \langle a_2, b^p \rangle \cong C_{p^m} \times C_p$ ,  $G' = \langle a_2, b^{p^m} \rangle \cong C_p^2$  and  $Z(G) = \langle x, b^p \rangle \cong C_{p^m} \times C_p$ .
- (I8)  $\langle a_1, x, b; a_2 \mid a_1^{p^2} = a_2^p = b^{p^m} = x^p = 1, [a_1, b] = a_2, [a_2, b] = a_1^{\nu p}, [a_2, a_1] = [x, a_1] = [x, b] = 1 \rangle = \langle a_1, b \rangle \times \langle x \rangle$ , where  $p > 2$  and  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ ; Moreover,  $|G| = p^{m+4}$ ,  $\Phi(G) = \langle a_2, a_1^p, b^p \rangle \cong C_{p^{m-1}} \times C_p^2$  if  $m > 1$ ,  $\Phi(G) \cong C_p^2$  if  $m = 1$ ,  $G' = \langle a_2, a_1^p \rangle \cong C_p^2$ ,  $Z(G) = \langle a_1^p, b^p, x \rangle \cong C_{p^{m-1}} \times C_p^2$  if  $m > 1$ ,  $Z(G) \cong C_p^2$  if  $m = 1$ .
- (I9)  $\langle a_1, x, b; a_2 \mid a_1^9 = a_2^3 = x^3 = 1, b^3 = a_1^3, [a_1, b] = a_2, [a_2, b] = a_1^{-3}, [x, a_1] = [x, a_2] = 1 \rangle = \langle a_1, b \rangle \times \langle x \rangle$ ; where  $|G| = 3^5$ ,  $\Phi(G) = G' = \langle a_2, a_1^3 \rangle \cong C_3^2$  and  $Z(G) = \langle a_1^3, x \rangle \cong C_3^2$ .
- (Iii)  $|G'| = p^3$  and  $c(G) = 3$ . In this case,  $Z(G) < \Phi(G)$ ,  $(\mu_0, \mu_1, \mu_2) = (1, 0, p^2 + p)$  and  $\alpha_1(G) = p^3 + p^2$ .
- (I10)  $\langle a, b, c \mid a^8 = b^{2^{m+1}} = 1, c^2 = a^4 b^{2^m}, [a, b] = a^2, [c, a] = 1, [c, b] = b^{2^m} \rangle$ ; where  $|G| = 2^{m+5}$ ,  $\Phi(G) = \langle a^2, b^2 \rangle \cong C_{2^m} \times C_4$ ,  $G' = \langle a^2, b^{2^m} \rangle \cong C_4 \times C_2$  and  $Z(G) = \langle a^4, b^2 \rangle \cong C_{2^m} \times C_2$ .
- (I11)  $\langle a, b, c; d \mid a^{p^{n+1}} = b^p = c^{p^2} = d^p = 1, [a, b] = d, [c, a] = a^{p^n}, [d, a] = c^p, [c, b] = [d, b] = [d, c] = 1 \rangle$ , where  $p > 2$ . Moreover,  $|G| = p^{n+5}$ ,  $\Phi(G) = \langle a^p, c^p, d \rangle \cong C_{p^n} \times C_p^2$ ,  $G' = \langle a^{p^n}, c^p, d \rangle \cong C_p^3$  and  $Z(G) = \langle c^p, a^p \rangle \cong C_{p^n} \times C_p$ .

**Proof** It is easy to see that  $Z(G) \not\leq \Phi(G)$  for groups (I1)–(I9) and  $Z(G) < \Phi(G)$  for (I10)–(I11). By checking maximal subgroups of groups (I1)–(I11), we know that they are all  $\mathcal{A}_3$ -groups and are pairwise non-isomorphic. Conversely, in the following, we prove that  $G$  is one of the groups (I1)–(I11).

Let  $A$  be an abelian subgroup of index  $p$  of  $G$ , and  $B$  be a non-abelian subgroup of index  $p$  of  $G$ . Then  $A \cap B$  is an abelian subgroup of index  $p$  of  $B$ . It follows that  $Z(B) \leq A \cap B$  and hence  $Z(B) \leq Z(G)$ .

We claim that there exists a non-abelian subgroup  $B$  of index  $p$  such that  $d(B) = 3$ . If not, then all non-abelian proper subgroups of  $G$  are generated by two elements. By Lemma 2.13,  $G \in \mathcal{A}_2$ , a contradiction.

By Lemma 2.6 (2),  $B' \leq C_p^2$  and  $B' \leq Z(B) \leq Z(G)$ . If  $G' = B'$ , then we have  $\Phi(G) \leq Z(G)$ , a contradiction. Hence  $G' > B'$ . By Lemma 2.15,  $|G'| \leq p|A'||B'| = p|B'|$ . Thus  $|G'| = p|B'|$ . Let  $D$  be another non-abelian subgroup of index  $p$  of  $G$  such that  $d(D) = 3$ . Similarly we have  $D' \leq C_p^2$  and  $D' \leq Z(D) \leq Z(G)$ . If  $D' \not\leq B'$ , then  $G' = B'D' \leq Z(G)$  and hence  $\Phi(G) \leq Z(G)$ , a contradiction. Thus we have  $D' \leq B'$ . The same reason gives that  $B' \leq D'$  and hence  $B' = D'$ . By arbitrariness of  $D$ , there exists a non-abelian subgroup  $M$  of index  $p$  such that  $d(M) = 2$  and all non-abelian subgroups of  $G/B'$  are generated by two elements.

Case 1.  $|G'| = p^2$

By Lemma 2.8,  $|G : Z(G)| = p^3$ . Since  $|G : \Phi(G)| = p^3$  and  $\Phi(G) \neq Z(G)$ , there exists a non-abelian subgroup  $M$  of index  $p$  such that  $Z(G) \not\leq M$ . Let  $x \in Z(G) \setminus M$ . Then  $G = \langle M, x \rangle$  and  $G' = M'$ . If  $d(M) = 3$ , then, by Lemma 2.6 (2),  $M' \leq C_p^3$  and  $M' \leq Z(M)$ . It follows that  $G' \leq C_p^3$  and  $G' \leq Z(G)$ . By calculation we get  $\Phi(G) \leq Z(G)$ , a contradiction. Hence  $d(M) = 2$ . It follows that  $\Phi(M) = \Phi(G)$  and  $M$  is one of the groups (1)–(7) in Lemma 2.5.

Subcase 1.1.  $M$  is the group (1) in Lemma 2.5. That is,  $M = \langle a, b \mid a^8 = b^{2^m} = 1, a^b = a^{-1} \rangle$ .

Since  $Z(M) = \langle a^4, b^2 \rangle$  and  $x^2 \in Z(G) \cap \Phi(G) = Z(M)$ , we may assume that  $x^2 = a^{4i}b^{2j}$ . Thus  $(xb^{-j})^2 = a^{4i}$ . If  $(2, j) = 1$ , then  $|\langle a^2, xb^{-j} \rangle| = 8$ . Since  $G$  is an  $\mathcal{A}_3$ -group, we have  $|G| \leq 2^5$  and  $|M| \leq 2^4$ . Hence  $m = 1$ . It follows that  $x^2 = a^{4i}$ . If  $2 \mid j$ , then, replacing  $x$  with  $xb^{-j}$ , we also have  $x^2 = a^{4i}$ . Hence we get the groups (I1) for  $2 \mid i$  and (I2) for  $(2, i) = 1$ , respectively.

Subcase 1.2.  $M$  is the group (2) in Lemma 2.5. That is,  $M = \langle a, b \mid a^8 = b^{2^m} = 1, a^b = a^3 \rangle$ .

Since  $Z(M) = \langle a^4, b^2 \rangle$  and  $x^2 \in Z(G) \cap \Phi(G) = Z(M)$ , we may assume that  $x^2 = a^{4i}b^{2j}$ . Thus  $(xb^{-j})^2 = a^{4i}$ . If  $(2, j) = 1$ , then  $|\langle a^2, xb^{-j} \rangle| = 8$ . Since  $G$  is an  $\mathcal{A}_3$ -group, we have  $|G| \leq 2^5$  and  $|M| \leq 2^4$ . Hence  $m = 1$ . It follows that  $x^2 = a^{4i}$ . If  $2 \mid j$ , then, replacing  $x$  with  $xb^{-j}$ , we also have  $x^2 = a^{4i}$ . If  $x^2 = 1$ , then we get the group (I3). If  $x^2 = a^4$ , then, replacing  $a$  with  $ax$ , we get the group (I2).

Subcase 1.3.  $M$  is the group (3) in Lemma 2.5. That is,  $M = \langle a, b \mid a^8 = 1, b^{2^m} = a^4, a^b = a^{-1} \rangle$ .

Since  $Z(M) = \langle a^4, b^2 \rangle$  and  $x^2 \in Z(G) \cap \Phi(G) = Z(M)$ , we may assume that  $x^2 = a^{4i}b^{2j}$ . Thus  $(xb^{-j})^2 = a^{4i}$ . If  $(2, j) = 1$ , then  $|\langle a^2, xb^{-j} \rangle| = 8$ . Since  $G$  is an  $\mathcal{A}_3$ -group, we have  $|G| \leq 2^5$  and  $|M| \leq 2^4$ . Hence  $m = 1$ . It follows that  $x^2 = a^{4(i+1)}$ . If  $2 \mid j$ , then, replacing  $x$  with  $xb^{-j}$ , we also have  $x^2 = a^{4i}$ . If  $x^2 = 1$ , then we get the group (I4). In the following, we may assume that  $x^2 = a^4$ . If  $m = 1$ , then, replacing  $b$  with  $xb$ , we get the group (I2). If  $m \geq 2$ , then, replacing  $x$  with  $xb^{2^{m-1}}$ , we get the group (I4).



Subcase 1.4.  $M$  is the group (4) in Lemma 2.5. That is,  $M = \langle a_1, b; a_2, a_3 \mid a_1^p = a_2^p = a_3^p = b^{p^m} = 1, [a_1, b] = a_2, [a_2, b] = a_3, [a_3, b] = 1, [a_i, a_j] = 1 \rangle$ , where  $p > 3$  for  $m = 1$ ,  $p > 2$  and  $1 \leq i, j \leq 3$ .

Since  $Z(M) = \langle a_3, b^p \rangle$  and  $x^p \in Z(G) \cap \Phi(G) = Z(M)$ , we may assume that  $x^p = a_3^i b^{jp}$ . Thus  $(xb^{-j})^p = a_3^i$ . If  $(j, p) = 1$ , then  $|\langle a_2, xb^{-j} \rangle| = p^3$ . Since  $G$  is an  $\mathcal{A}_3$ -group, we have  $|G| \leq p^5$  and  $|M| \leq p^4$ . Hence  $m = 1$ . It follows that  $x^p = a_3^i$ . If  $p \mid j$ , then, replacing  $x$  with  $xb^{-j}$ , we also have  $x^p = a_3^i$ . If  $p \mid i$ , then we get the group (I5). If  $(i, p) = 1$ , then, replacing  $x$  with  $x^{i^{-1}}$ , we get the group (I6).

Subcase 1.5.  $M$  is the group (5) in Lemma 2.5. That is,  $M = \langle a_1, b; a_2 \mid a_1^p = a_2^p = b^{p^{m+1}} = 1, [a_1, b] = a_2, [a_2, b] = b^{p^m}, [a_1, a_2] = 1 \rangle$ , where  $p > 2$ .

Since  $Z(M) = \langle b^p \rangle$  and  $x^p \in Z(G) \cap \Phi(G) = Z(M)$ , we may assume that  $x^p = b^{jp}$ . Thus  $(x^{-j^{-1}}b)^p = 1$ . If  $p \mid j$ , then, replacing  $x$  with  $x^{-j^{-1}}b$ , we get the group (I7). If  $(j, p) = 1$ , then  $|\langle a_2, x^{-j^{-1}}b \rangle| = p^3$ . Since  $G$  is an  $\mathcal{A}_3$ -group, we have  $|G| \leq p^5$  and  $|M| \leq p^4$ . Hence  $m = 1$ . By replacing  $b$  with  $x^{-j^{-1}}b$  we get the group (I6).

Subcase 1.6.  $M$  is the group (6) in Lemma 2.5. That is,  $M = \langle a_1, b; a_2 \mid a_1^{p^2} = a_2^p = b^{p^m} = 1, [a_1, b] = a_2, [a_2, b] = a_1^{\nu p}, [a_1, a_2] = 1 \rangle$ , where  $p > 2$  and  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ .

Since  $Z(M) = \langle a_1^p, b^p \rangle$  and  $x^p \in Z(G) \cap \Phi(G) = Z(M)$ , we may assume that  $x^p = a_1^{ip} b^{jp}$ . Thus  $(x^{-j^{-1}}b)^p = a_1^{ip}$ . If  $(j, p) = 1$ , then  $|\langle a_2, x^{-j^{-1}}b \rangle| = p^3$ . Since  $G$  is an  $\mathcal{A}_3$ -group, we have  $|G| \leq p^5$  and  $|M| \leq p^4$ . Hence  $m = 1$ . It follows that  $x^p = a_1^{ip}$ . If  $p \mid j$ , then, replacing  $x$  with  $xb^{-j}$ , we also have  $x^p = a_1^{ip}$ . If  $p \mid i$ , then we get the group (I8). If  $(i, p) = 1$ , then, replacing  $a_1$  and  $x$  with  $a_1 x^{-i^{-1}}$  and  $x^{i^{-1}\nu}$  respectively, we get the group (I6).

Subcase 1.7.  $M$  is the group (7) in Lemma 2.5. That is,  $M = \langle a_1, b; a_2 \mid a_1^9 = a_2^3 = 1, b^3 = a_1^3, [a_1, b] = a_2, [a_2, b] = a_1^{-3} \rangle$ .

If  $x^3 = 1$ , then  $G$  is the group (I9). If  $x^3 \neq 1$ , then we may assume that  $a_1^3 = b^3 = x^{-3}$ . By replacing  $a_1$  and  $b$  with  $a_1 x$  and  $bx$  respectively, we get the group (I6).

Case 2.  $|G'| = p^3$

In this case,  $B' \cong C_p^2$  and  $Z(B) = \Phi(B)$ . By Lemma 2.8,  $|G : Z(G)| = p^4$ . Since  $|G : Z(B)| = p^4$  and  $Z(B) \leq Z(G)$ , we have  $Z(G) = Z(B) = \Phi(B) < \Phi(G)$ . By Lemma 2.13,  $G/B' \in \mathcal{A}_2$ . Hence  $G/B'$  is one of the groups (8)–(12) in Lemma 2.5.

Subcase 2.1.  $G/B'$  is the group (8) or (11) in Lemma 2.5. That is,  $G/B' = \langle \bar{a}, \bar{b}, \bar{c} \mid \bar{a}^4 = \bar{c}^2 = 1, \bar{b}^2 = \bar{a}^2 = [\bar{a}, \bar{b}], [\bar{c}, \bar{a}] = [\bar{c}, \bar{b}] = 1 \rangle \cong Q_8 \times C_2$  or  $G/B' = \langle \bar{a}, \bar{b}, \bar{c} \mid \bar{a}^4 = 1, \bar{b}^2 = \bar{c}^2 = \bar{a}^2 = [\bar{a}, \bar{b}], [\bar{c}, \bar{a}] = [\bar{c}, \bar{b}] = 1 \rangle \cong Q_8 * C_4$ .

In this subcase,  $\Phi(G) = G' = \langle a^2, B' \rangle$ . Since  $a^2 = b^2 z$  where  $z \in B' \leq Z(G)$ , we have  $[a^2, b] = 1$ . It follows that  $\Phi(G) \leq Z(G)$ , a contradiction.

Subcase 2.2.  $G/B'$  is the group (9) in Lemma 2.5. That is,  $G/B' = \langle \bar{a}, \bar{b}, \bar{c} \mid \bar{a}^{p^{n+1}} = \bar{b}^{p^m} = \bar{c}^p = 1, [\bar{a}, \bar{b}] = \bar{a}^{p^n}, [\bar{c}, \bar{a}] = [\bar{c}, \bar{b}] = 1 \rangle \cong M(n+1, m) \times C_p$ .

If  $o(a) = p^{n+1}$ , then, by calculation, we have  $\Phi(G) \leq Z(G)$ , a contradiction. Hence  $o(a) = p^{n+2}$ . If  $p > 2$ , then  $\langle a, b \rangle$  has no abelian subgroup of index  $p$ , a contradiction. Hence  $p = 2$ . If  $b^{2^m} \in \langle a^{2^{n+1}} \rangle$ , then  $\langle a^2, b \rangle$  is non-abelian and is of order  $2^{n+m+1}$ , which is contrary to  $G \in \mathcal{A}_3$ . Hence  $o(b) = 2^{m+1}$  and  $B' = \langle a^{2^{n+1}}, b^{2^m} \rangle$ . Since  $\langle a, b \rangle$  has an abelian subgroup of index  $p$ , by calculation we have  $n = 1$  and  $A \cap \langle a, b \rangle = \langle a, b^2 \rangle$ , where  $A$  is the abelian subgroup of index  $p$  of  $G$ . Hence  $A = \langle a, b^2, c \rangle$ . It follows that  $[c, a] = 1$ . Assume that  $[c, b] = a^{4w} b^{u2^m}$ . By replacing  $c$  with  $ca^{2w}$  we have  $[c, b] = b^{u2^m}$ . Since  $Z(G) < \Phi(G)$ , we have  $[c, b] = b^{2^m}$ . If  $c^2 \in \langle b^{2^m} \rangle$ , then  $|\langle c, b \rangle| = 2^{m+2}$ , which is contrary to  $G \in \mathcal{A}_3$ . Thus we may assume that

$$G = \langle a, b, c \mid a^8 = b^{2^{m+1}} = 1, c^2 = a^4 b^{i2^m}, [a, b] = a^2 a^{4j} b^{k2^m}, [c, a] = 1, [c, b] = b^{2^m} \rangle.$$

By replacing  $a$  with  $ac^j$  we may assume that  $[a, b] = a^2 b^{k'2^m}$ . If  $m \geq 2$ , then, replacing  $a$  and  $c$  with  $ab^{k'2^{m-1}}$  and  $cb^{(i+1)2^{m-1}}$  respectively, we have  $[a, b] = a^2$  and  $c^2 = a^4 b^{2^m}$ . Hence we get the group (I10). If  $m = 1$ , then  $(ba)^2 = b^2 a^2 [a, b] = a^4 b^{2(k'+1)}$ . Since  $G \in \mathcal{A}_3$ , we have  $|\langle a^2, ba \rangle| = 2^4$ . It follows that  $(ba)^2 = a^4 b^2$  and  $[a, b] = a^2$ . Again since  $G \in \mathcal{A}_3$ , we have  $|\langle ca^2, ba \rangle| = 2^4$ . It follows that  $(ca)^2 = b^2$  and  $c^2 = a^4 b^2$ . Hence we also get the group (I10).

Subcase 2.3.  $G/B'$  is the group (10) in Lemma 2.5. That is,  $G/B' = \langle \bar{a}, \bar{b}, \bar{c}; \bar{d} \mid \bar{a}^{p^n} = \bar{b}^{p^m} = \bar{c}^p = \bar{d}^p = 1, [\bar{a}, \bar{b}] = \bar{d}, [\bar{c}, \bar{a}] = [\bar{c}, \bar{b}] = 1 \rangle \cong M(n, m, 1) \times C_p$ , where  $n \geq m$ , and  $n \geq 2$  if  $p = 2$ .

Since  $\langle \bar{a}, \bar{b} \rangle$  is not metacyclic,  $\langle a, b \rangle$  is not metacyclic. Since  $\langle a, b \rangle$  is a two-generator  $\mathcal{A}_2$ -subgroup, by Lemma 2.6 (6) and (7) we have  $p > 2$  and  $\exp(\langle a, b \rangle') = p$ . Hence  $d^p = 1$  and we may assume  $[a, b] = d$ . If  $m \geq 2$ , then

$$|G : \langle d, a \rangle| \geq p^{m+1} \geq p^3 \text{ and } |G : \langle d, b \rangle| \geq p^{n+1} \geq p^3.$$

Since  $G \in \mathcal{A}_3$ ,  $[d, a] = [d, b] = 1$ . It follows that  $\langle a, b \rangle \in \mathcal{A}_1$ , a contradiction. Hence we have  $m = 1$ . Since the index of  $\langle b, c \rangle$  is at least  $p^{n+1}$ , we have  $[b, c] = 1$  for  $n \geq 2$ . If  $n = 1$ , then, without loss of generality, we may assume  $[b, c] = 1$ . Since  $Z(G) < \Phi(G)$ , we have  $[c, a] \neq 1$ . Hence  $A = \langle b, c, \Phi(G) \rangle$ . It follows that  $[d, b] = 1$  and hence  $[d, a] \neq 1$ . Since  $G \in \mathcal{A}_3$  and  $|G| = p^{n+5}$ , we have  $|\langle a, d \rangle| = p^{n+3}$ . Hence

$$o(a) = p^{n+1}, [d, a] \notin \langle a^{p^n} \rangle \text{ and } B' = \langle a^{p^n}, [d, a] \rangle.$$

Since  $Z(G) < \Phi(G)$ , we have  $[c, a] \notin \langle [d, a] \rangle$ . Hence we may assume  $[c, a] = a^{ip^n} [d, a]^j$  where  $(i, p) = 1$ . By replacing  $c$  with  $c^{i^{-1}} d^{-i^{-1}j}$  we have  $[c, a] = a^{p^n}$ . Hence  $|\langle c, a \rangle| = p^{n+3}$  and  $c^p \notin \langle a^{p^n} \rangle$ . It follows that  $G' = \langle d, a^{p^n}, c^p \rangle$ . Hence we may assume  $[d, a] = c^{up} a^{vp^n}$  where  $(u, p) = 1$ . Since  $|\langle d, c^u a^{vp^{n-1}} \rangle| = p^3 \leq p^{n+3}$ ,  $[d, c^u a^{vp^{n-1}}] = 1$  and hence

$n \geq 2$  or  $v = 0$ . By replacing  $b, c$  and  $d$  with  $b^{u^{-1}}, ca^{u^{-1}vp^{n-1}}$  and  $d^{u^{-1}}$ , respectively, we get  $[d, a] = c^p$ . We may assume  $b^p = a^{sp^n} \pmod{\langle c^p \rangle}$ . By calculation we have  $(ba^{-sp^{n-1}})^p \in \langle c^p \rangle$ . Since  $|\langle c, ba^{-sp^{n-1}} \rangle| \leq p^3$ , we have  $[c, ba^{-sp^{n-1}}] = 1$ . Assume  $(ba^{-sp^{n-1}})^p = c^{tp}$ . Since  $|\langle d, ba^{-sp^{n-1}}c^{-t} \rangle| \leq p^3$ , we have  $[d, ba^{-sp^{n-1}}c^{-t}] = 1$ . By replacing  $b$  with  $ba^{-sp^{n-1}}c^{-t}$  we have  $b^p = 1$ . Hence  $G$  is the group (I11).

Subcase 2.4.  $G/B'$  is the group (12) in Lemma 2.5. That is,  $G/B' = \langle \bar{a}, \bar{b}, \bar{c} \mid \bar{a}^{p^n} = \bar{b}^{p^m} = \bar{c}^{p^2} = 1, [\bar{a}, \bar{b}] = \bar{c}^p, [\bar{c}, \bar{a}] = [\bar{c}, \bar{b}] = 1 \rangle \cong M(n, m, 1) * C_{p^2}$ , where  $n \geq m$ , and  $n \geq 2$  if  $p = 2$ .

Since  $\langle \bar{a}, \bar{b} \rangle$  is not metacyclic,  $\langle a, b \rangle$  is not metacyclic. Since  $\langle a, b \rangle$  is a two-generator  $\mathcal{A}_2$ -group, by Lemma 2.6 (6) and (7) we have  $p > 2$  and  $\exp(\langle a, b \rangle') = p$ . Hence  $c^{p^2} = 1$  and  $\exp(G') = p$ . It follows that  $[c^p, a] = [c^p, b] = 1$ . Thus  $\langle a, b \rangle \in \mathcal{A}_1$ , a contradiction.

We calculate the  $(\mu_0, \mu_1, \mu_2)$  and  $\alpha_1(G)$  of those groups in Theorem 5.4 as follows. It is easy to see that  $(\mu_0, \mu_1, \mu_2) = (1, 0, p^2 + p)$ . We only need to calculate  $\alpha_1(G)$ .

**Case 1.**  $G$  is one of the groups (I1)–(I4).

All maximal subgroups of  $G$  are:

$$N = \langle a, x, \Phi(G) \rangle;$$

$$N_i = \langle ba^i, x, \Phi(G) \rangle, \text{ where } i = 0, 1;$$

$$N_{ij} = \langle ax^i, bx^j, \Phi(G) \rangle, \text{ where } i, j = 0, 1.$$

It is easy to see that  $N = \langle a, x, \Phi(G) \rangle$  is the unique abelian maximal subgroup of  $G$ . Since  $|N'_i| = p$  and  $d(N_i) = 3$ , by Lemma 2.6 (7),  $\alpha_1(N_i) = p^2 = 4$ . Since  $N_{ij} = \langle ax^i, bx^j \rangle$ ,  $\alpha_1(N_{ij}) = p = 2$ .

Let  $H \in \Gamma_2(G)$ . Then  $H = \langle a^i b^j x^k, \Phi(G) \rangle$  where  $(i, j, k) \neq (0, 0, 0)$ . It is obvious that  $H \in \mathcal{A}_1$  if and only if  $j \neq 0$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) - p \sum_{H \in \Gamma_2} \alpha_1(H) = p \times p^2 + p^2 \times p - p \times p^2 = p^3.$$

**Case 2.**  $G$  is one of the groups (I5)–(I9).

All maximal subgroups of  $G$  are:

$$N = \langle a_1, x, \Phi(G) \rangle;$$

$$N_i = \langle ba_1^i, x, \Phi(G) \rangle, \text{ where } 0 \leq i \leq p-1;$$

$$N_{ij} = \langle a_1 x^i, b x^j, \Phi(G) \rangle, \text{ where } 0 \leq i, j \leq p-1.$$

It is easy to see that  $N = \langle a_1, x, \Phi(G) \rangle$  is the unique abelian maximal subgroup of  $G$ . Since  $|N'_i| = p$  and  $d(N_i) = 3$ , by Lemma 2.6,  $\alpha_1(N_i) = p^2$ . Since  $N_{ij} = \langle a_1 x^i, b x^j \rangle$ ,  $\alpha_1(N_{ij}) = p$ .

Let  $H \in \Gamma_2(G)$ . Then  $H = \langle a_1^i b^j x^k, \Phi(M) \rangle$  where  $(i, j, k) \neq (0, 0, 0)$ . It is obvious that  $H \in \mathcal{A}_1$  if and only if  $j \neq 0$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) - p \sum_{H \in \Gamma_2} \alpha_1(H) = p \times p^2 + p^2 \times p - p \times p^2 = p^3.$$

**Case 3.**  $G$  is the group (I10).

All maximal subgroups of  $G$  are:

$$N = \langle a, c, \Phi(G) \rangle;$$

$$N_i = \langle ba^i, c, \Phi(G) \rangle, \text{ where } i = 0, 1;$$

$$N_{ij} = \langle ac^i, bc^j, \Phi(G) \rangle, \text{ where } i, j = 0, 1.$$

It is easy to see that  $N = \langle a, c, \Phi(G) \rangle$  is the unique abelian maximal subgroup of  $G$ . Since  $|N'_i| = p^2 = 4$  and  $d(N_i) = 3$ , by Lemma 2.6,  $\alpha_1(N_i) = p^2 + p$ . By calculation,  $N_{ij} = \langle ac^i, bc^j \rangle$ . Hence  $\alpha_1(N_{ij}) = p$ .

Let  $H \in \Gamma_2(G)$ . Then  $H = \langle a^i b^j c^k, \Phi(G) \rangle$  where  $(i, j, k) \neq (0, 0, 0)$ . It is obvious that  $H \in \mathcal{A}_1$  if and only if  $j \neq 0$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) - p \sum_{H \in \Gamma_2} \alpha_1(H) = p \times (p^2 + p) + p^2 \times p - p \times p^2 = p^3 + p^2.$$

**Case 4.**  $G$  is the group (I11).

All maximal subgroups of  $G$  are:

$$N = \langle b, c, \Phi(G) \rangle;$$

$$N_i = \langle ab^i, c, \Phi(G) \rangle, \text{ where } 0 \leq i \leq p-1;$$

$$N_{ij} = \langle ac^i, bc^j, \Phi(G) \rangle, \text{ where } 0 \leq i, j \leq p-1.$$

It is easy to see that  $N = \langle b, c, \Phi(G) \rangle$  is the unique abelian maximal subgroup of  $G$ . Since  $|N'_i| = p^2$  and  $d(N_i) = 3$ , by Lemma 2.6,  $\alpha_1(N_i) = p^2 + p$ . By calculation,  $N_{ij} = \langle ac^i, bc^j \rangle$ . Hence  $\alpha_1(N_{ij}) = p$ .

Let  $H \in \Gamma_2$ . Then  $H = \langle a^i b^j c^k, \Phi(G) \rangle$  where  $(i, j, k) \neq (0, 0, 0)$ . It is obvious that  $H \in \mathcal{A}_1$  if and only if  $i \neq 0$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) - p \sum_{H \in \Gamma_2} \alpha_1(H) = p \times (p^2 + p) + p^2 \times p - p \times p^2 = p^3 + p^2.$$

□

**Theorem 5.5.**  $d(G) = 4$  if and only if  $G$  is isomorphic to one of the following pairwise non-isomorphic groups:

(Ji)  $G' \cong C_p$  and  $c(G) = 2$ . In this case,  $(\mu_0, \mu_1, \mu_2) = (1 + p, 0, p^2 + p^3)$  and  $\alpha_1(G) = p^4$ .

(J1)  $\langle a, b, x, y \mid a^4 = x^2 = y^2 = 1, b^2 = a^2 = [a, b], [x, a] = [x, b] = [y, a] = [y, b] = [x, y] = 1 \rangle \cong Q_8 \times C_2 \times C_2$ ; where  $|G| = 2^5$ ,  $\Phi(G) = G' = \langle a^2 \rangle$  and  $Z(G) = \langle a^2, x, y \rangle \cong C_2^3$ .

(J2)  $\langle a, b, x, y \mid a^{p^{n+1}} = b^{p^m} = x^p = y^p = 1, [a, b] = a^{p^n}, [x, a] = [x, b] = [y, a] = [y, b] = [x, y] = 1 \rangle \cong M_p(n+1, m) \times C_p \times C_p$ ; where  $|G| = p^{n+m+3}$ ,  $\Phi(G) = \langle a^p, b^p \rangle \cong C_{p^n} \times C_{p^{m-1}}$  if  $m > 1$ ,  $\Phi(G) \cong C_{p^n}$  if  $m = 1$ ,  $G' = \langle a^{p^n} \rangle$ ,  $Z(G) = \langle a^p, b^p, x, y \rangle \cong C_{p^n} \times C_{p^{m-1}} \times C_p^2$  if  $m > 1$ ,  $Z(G) \cong C_{p^n} \times C_p^2$  if  $m = 1$ .

- (J3)  $\langle a, b, x, y; c \mid a^{p^n} = b^{p^m} = c^p = x^p = y^p = 1, [a, b] = c, [c, a] = [c, b] = [x, a] = [x, b] = [y, a] = [y, b] = [x, y] = 1 \rangle \cong M_p(n, m, 1) \times C_p \times C_p$ , where  $n \geq m$ , and  $n \geq 2$  if  $p = 2$ ; moreover,  $|G| = p^{n+m+3}$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^{n-1}} \times C_{p^{m-1}} \times C_p$  if  $m > 1$ ,  $\Phi(G) \cong C_{p^{n-1}} \times C_p$  if  $m = 1$  and  $n > 1$ ,  $\Phi(G) \cong C_p$  if  $m = n = 1$ ,  $G' = \langle c \rangle$ ,  $Z(G) = \langle a^p, b^p, c, x, y \rangle \cong C_{p^{n-1}} \times C_{p^{m-1}} \times C_p^3$  if  $m > 1$ ,  $Z(G) \cong C_{p^{n-1}} \times C_p^3$  if  $m = 1$  and  $n > 1$ ,  $Z(G) \cong C_p^3$  if  $m = n = 1$ .
- (J4)  $\langle a, b, x, y \mid a^4 = y^2 = 1, b^2 = x^2 = a^2 = [a, b], [x, a] = [x, b] = [y, a] = [y, b] = [x, y] = 1 \rangle \cong Q_8 * C_4 \times C_2$ ; where  $|G| = 2^5$ ,  $\Phi(G) = G' = \langle a^2 \rangle$  and  $Z(G) = \langle x, y \rangle \cong C_4 \times C_2$ .
- (J5)  $\langle a, b, x, y \mid a^{p^n} = b^{p^m} = x^{p^2} = y^p = 1, [a, b] = x^p, [x, a] = [x, b] = [y, a] = [y, b] = [x, y] = 1 \rangle \cong M_p(n, m, 1) * C_{p^2} \times C_p$ , where  $n \geq 2$  if  $p = 2$  and  $n \geq m$ . Moreover,  $|G| = p^{n+m+3}$ ,  $\Phi(G) = \langle a^p, b^p, x^p \rangle \cong C_{p^{n-1}} \times C_{p^{m-1}} \times C_p$  if  $m > 1$ ,  $\Phi(G) \cong C_{p^{n-1}} \times C_p$  if  $m = 1$  and  $n > 1$ ,  $\Phi(G) \cong C_p$  if  $m = n = 1$ ,  $G' = \langle x^p \rangle$ ,  $Z(G) = \langle a^p, b^p, x, y \rangle \cong C_{p^{n-1}} \times C_{p^{m-1}} \times C_p \times C_{p^2}$  if  $m > 1$ ,  $Z(G) \cong C_{p^{n-1}} \times C_p \times C_{p^2}$  if  $m = 1$  and  $n > 1$ ,  $Z(G) \cong C_p \times C_{p^2}$  if  $m = n = 1$ .
- (Jii)  $G' \cong C_p^2$  and  $c(G) = 2$ . In this case,  $(\mu_0, \mu_1, \mu_2) = (1, 0, p + p^2 + p^3)$  and  $\alpha_1(G) = p^4 + p^3$ .
- (J6)  $K \times C_2$ , where  $K = \langle a, b, c \mid a^4 = b^4 = 1, c^2 = a^2 b^2, [a, b] = b^2, [c, a] = a^2, [c, b] = 1 \rangle$ ; and  $|G| = 2^6$ ,  $\Phi(G) = G' = \langle a^2, b^2 \rangle \cong C_2^2$ ,  $Z(G) \cong C_2^3$ .
- (J7)  $K \times C_p$ , where  $K = \langle a, b, d \mid a^{p^m} = b^{p^2} = d^p = 1, [a, b] = a^{p^{m-1}}, [d, a] = b^p, [d, b] = 1 \rangle$ , where  $m \geq 3$  if  $p = 2$ ; moreover,  $|G| = p^{m+4}$ ,  $\Phi(G) = \langle a^p, b^p \rangle \cong C_{p^{m-1}} \times C_p$ ,  $G' = \langle a^{p^{m-1}}, b^p \rangle$ ,  $Z(G) \cong C_{p^{m-1}} \times C_p^2$ .
- (J8)  $K \times C_p$ , where  $K = \langle a, b, d \mid a^{p^m} = b^{p^2} = d^{p^2} = 1, [a, b] = d^p, [d, a] = b^{jp}, [d, b] = 1 \rangle$ , where  $(j, p) = 1$ ,  $p > 2$ ,  $j$  is a fixed quadratic non-residue modulo  $p$ , and  $-4j$  is a quadratic non-residue modulo  $p$ ; moreover,  $|G| = p^{m+5}$ ,  $\Phi(G) = \langle a^p, b^p, d^p \rangle \cong C_{p^{m-1}} \times C_p^2$  if  $m > 1$ ,  $\Phi(G) \cong C_p^2$  if  $m = 1$ ,  $G' = \langle b^p, d^p \rangle$ ,  $Z(G) \cong C_{p^{m-1}} \times C_p^3$  if  $m > 1$ ,  $Z(G) \cong C_p^3$  if  $m = 1$ .
- (J9)  $K \times C_p$ , where  $K = \langle a, b, d \mid a^{p^m} = b^{p^2} = d^{p^2} = 1, [a, b] = d^p, [d, a] = b^{jp} d^p, [d, b] = 1 \rangle$ , where if  $p$  is odd, then  $4j = 1 - \rho^{2r+1}$  with  $1 \leq r \leq \frac{p-1}{2}$  and  $\rho$  the smallest positive integer which is a primitive root (mod  $p$ ); if  $p = 2$ , then  $j = 1$ . Moreover,  $|G| = p^{m+5}$ ,  $\Phi(G) = \langle a^p, b^p, d^p \rangle \cong C_{p^{m-1}} \times C_p^2$  if  $m > 1$ ,  $\Phi(G) \cong C_p^2$  if  $m = 1$ ,  $G' = \langle b^p, d^p \rangle$ ,  $Z(G) \cong C_{p^{m-1}} \times C_p^3$  if  $m > 1$ ,  $Z(G) \cong C_p^3$  if  $m = 1$ .

**Proof** By Lemma 3.5, groups (J1)–(J9) are  $\mathcal{A}_3$ -groups. By checking maximal subgroups of groups (J1)–(J9), we know that they are pairwise non-isomorphic. Conversely, in the following, we prove that  $G$  is one of the groups (J1)–(J9).

By Lemma 3.17, we have  $c(G) = 2$ ,  $\Phi(G) \leq Z(G)$ , all  $\mathcal{A}_1$ -subgroups of  $G$  contain  $\Phi(G)$  and  $G' \leq C_p^3$ . Let  $S$  be an abelian subgroup of index  $p$  of  $G$  and  $t \in G \setminus A$ . By Lemma 2.8,  $G' = \langle [s, t] \mid s \in S \rangle$ .

We claim  $G' \leq C_p^2$ . Otherwise,  $G' \cong C_p^3$ . Let  $M = \langle s, t \rangle$  be an  $\mathcal{A}_1$ -subgroup of  $G$ . Since  $G' \leq M$ ,  $M$  is not metacyclic. Hence we may assume that

$$M = \langle s, t \mid s^{p^m} = t^{p^n} = r^p = 1, [s, t] = r, [r, s] = [r, t] = 1 \rangle$$

and

$$G' = \langle s^{p^{m-1}}, t^{p^{n-1}}, r \rangle.$$

By Lemma 2.8, there exists  $a \in S$  such that  $[a, t] = t^{p^{n-1}}$ . Hence  $\langle a, t \rangle$  is metacyclic, which is contrary to  $G' \leq \langle a, t \rangle$ . Hence  $G' \leq C_p^2$ .

By Lemma 2.16, there exists  $K \leq G$  such that  $K' = G'$  and  $d(K) = 3$ . Since  $K \in \mathcal{A}_2$ ,  $K$  is maximal in  $G$ . By Lemma 2.8,

$$|Z(G)| = \frac{|G|}{p|G'|} \text{ and } |Z(K)| = \frac{|K|}{p|K'|}.$$

It follows that  $Z(G) \not\leq K$ . Let  $y \in Z(G) \setminus K$ . Then  $G = \langle K, y \rangle$ . By Lemma 2.5,  $K$  is one of the groups (8)–(16) in Lemma 2.5.

Case 1.  $K$  is one of the groups (8)–(10) in Lemma 2.5. That is,  $K = \langle a, b \rangle \times \langle x \rangle$ .

Let  $L = \langle a, b, y \rangle$ . Then  $G = L \times \langle x \rangle$  and  $L$  is a group of Type (8)–(12) in Lemma 2.5. Hence we get groups (J1)–(J5).

Case 2.  $K$  is the groups (11) in Lemma 2.5. That is,  $K = \langle a, b, x \mid a^4 = 1, b^2 = x^2 = a^2 = [a, b], [x, a] = [x, b] = 1 \rangle \cong Q_8 * C_4$ .

Since  $y^2 \in \Phi(G) = \Phi(K) = \langle x^2 \rangle$ , we may assume  $y^2 = x^{2i}$ . By replacing  $y$  with  $yx^i$  we have  $y^2 = 1$ . Hence we get the group (J4).

Case 3.  $K$  is the group (12) in Lemma 2.5. That is,  $K = \langle a, b, x \mid a^{p^n} = b^{p^m} = x^{p^2} = 1, [a, b] = x^p, [x, a] = [x, b] = 1 \rangle \cong M_p(n, m, 1) * C_{p^2}$ , where  $n \geq 2$  if  $p = 2$  and  $n \geq m$ .

Let  $L = \langle a, b, y \rangle$ . Then  $G = L * \langle x \rangle$  and  $L$  is one of the groups (8)–(12) in Lemma 2.5. If  $L$  is one of the groups (8)–(11) in Lemma 2.5, then it is reduced to Case 1 or 2. Hence we may assume that  $L$  is also a group of Type (12) in Lemma 2.5. Let

$$L = \langle a, b, y \mid a^{p^n} = b^{p^m} = y^{p^2} = 1, [a, b] = y^p, [y, a] = [y, b] = 1 \rangle.$$

Since  $x^p \in G' = K'$ , we may assume that  $x^p = y^{ip}$ . By replacing  $x$  with  $xy^{-i}$  we have  $x^p = 1$ . Hence we get the group (J5).

Case 4.  $K$  is the group (13) in Lemma 2.5. That is,  $K = \langle a, b, c \mid a^4 = b^4 = 1, c^2 = a^2b^2, [a, b] = b^2, [c, a] = a^2, [c, b] = 1 \rangle$ .

Since  $y^2 \in \Phi(G) = \Phi(K) = \langle a^2, b^2 \rangle$ , we may assume  $y^2 = a^{2i}b^{2j}$ . Since  $|\langle ya^i, b \rangle| \leq 8$ , we have  $[ya^i, b] = 1$  and hence  $y^2 = b^{2j} = c^{2j}a^{2j}$ . Since  $|\langle yc^j, a \rangle| \leq 8$ , we have  $[yc^j, a] = 1$  and hence  $y^2 = 1$ . Thus we get the group (J6).

Case 5.  $K$  is the group (14) in Lemma 2.5. That is,  $K = \langle a, b, d \mid a^{p^m} = b^{p^2} = d^p = 1, [a, b] = a^{p^{m-1}}, [d, a] = b^p, [d, b] = 1 \rangle$ , where  $m \geq 3$  if  $p = 2$ .

Since  $y^p \in \Phi(G) = \Phi(K) = \langle a^p, b^p \rangle$ , we may assume  $y^p = a^{ip}b^{jp}$ . Since  $|\langle yb^{-j}, a \rangle| \leq p^{m+1}$ , we have  $[yb^{-j}, a] = 1$  and hence  $y^p = a^{ip}$ . Since  $|\langle ya^{-i}, d \rangle| \leq p^3$ , we have  $[ya^{-i}, d] = 1$  and hence  $a^i \in Z(K)$ . By replacing  $y$  with  $ya^{-i}$  we have  $y^p = 1$ . Hence we get the group (J7).

Case 6.  $K$  is the group (15) in Lemma 2.5. That is,  $K = \langle a, b, d \mid a^{p^m} = b^{p^2} = d^{p^2} = 1, [a, b] = d^p, [d, a] = b^{jp}, [d, b] = 1 \rangle$ , where  $(j, p) = 1$ ,  $p > 2$ ,  $j$  is a fixed quadratic non-residue modulo  $p$ , and  $-4j$  is a quadratic non-residue modulo  $p$ .

Since  $y^p \in \Phi(G) = \Phi(K) = \langle b^p, d^p \rangle$ , we may assume that  $y^p = b^{rp}d^{sp}$ . Since  $|\langle yb^{-r}, a \rangle| \leq p^{m+2}$ , we have  $[yb^{-r}, a] = 1$  and hence  $y^p = d^{sp}$ . Since  $|\langle yd^{-s}, a \rangle| \leq p^{m+2}$ , we have  $[yd^{-s}, a] = 1$  and hence  $y^p = 1$ . Thus we get the group (J8).

Case 7.  $K$  is the group (16) in Lemma 2.5. That is,  $K = \langle a, b, d \mid a^{p^m} = b^{p^2} = d^{p^2} = 1, [a, b] = d^p, [d, a] = b^{jp}d^p, [d, b] = 1 \rangle$ , where if  $p$  is odd, then  $4j = 1 - \rho^{2r+1}$  with  $1 \leq r \leq \frac{p-1}{2}$  and  $\rho$  the smallest positive integer which is a primitive root (mod  $p$ ); if  $p = 2$ , then  $j = 1$ .

Since  $y^p \in \Phi(G) = \Phi(K) = \langle b^p, d^p \rangle$ , we may assume  $y^p = b^{rp}d^{sp}$ . Since  $|\langle yb^{-r}, a \rangle| \leq p^{m+2}$ , we have  $[yb^{-r}, a] = 1$  and hence  $y^p = d^{sp}$ . Since  $|\langle yd^{-s}, a \rangle| \leq p^{m+2}$ , we have  $[yd^{-s}, a] = 1$  and hence  $y^p = 1$ . Thus we get the group (J9).

We calculate the  $(\mu_0, \mu_1, \mu_2)$  and  $\alpha_1(G)$  of those groups in Theorem 5.5 as follows.

Since  $d(G) = 4$  and  $\mu_1 = 0$ ,  $\mu_0 + \mu_2 = 1 + p + p^2 + p^3$ . Let  $N$  be an  $\mathcal{A}_1$ -subgroup of  $G$ . Since  $G \in \mathcal{A}_3$ ,  $N\Phi(G) \in \Gamma_1$  or  $\Gamma_2$ . If  $N\Phi(G) \in \Gamma_1$ , then  $d(G) \leq 3$ , a contradiction. Hence  $N\Phi(G) \in \Gamma_2$ . Since  $G \in \mathcal{A}_3$ ,  $N\Phi(G) \in \mathcal{A}_1$ . That is,  $N = N\Phi(G) \in \Gamma_2$ . Hence  $\sum_{H \in \Gamma_2} \alpha_1(H) = \alpha_1(G)$ . By Hall's enumeration principle,  $\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) - p \sum_{H \in \Gamma_2} \alpha_1(H) = \sum_{H \in \Gamma_1} \alpha_1(H) - p\alpha_1(G)$ . Hence  $\alpha_1(G) = \frac{1}{1+p} \sum_{H \in \Gamma_1} \alpha_1(H)$ .

**Case 1.**  $G$  is one of the groups (J1)–(J5).

Since  $Z(G)$  is of index  $p^2$ ,  $\mu_0 = 1 + p$  and hence  $\mu_2(G) = p^2 + p^3$ . Let  $H \in \Gamma_1$ . If  $H$  is not abelian, then  $|H'| = p$ . By Lemma 2.6,  $\alpha_1(H) = p^2$ . Hence

$$\alpha_1(G) = \frac{1}{1+p} \sum_{H \in \Gamma_1} \alpha_1(H) = \frac{1}{1+p} \mu_2 p^2 = p^4.$$

**Case 2.**  $G$  is one of the groups (J6)–(J9).

Let  $A$  be the unique abelian maximal subgroup of  $K$ . Then  $A \times \langle x \rangle$  is the unique abelian maximal subgroup of  $G$ . Other maximal subgroups of  $G$  are:

$N_i = M_i \times \langle x \rangle$ , where  $M_i$  are non-abelian maximal subgroups of  $K$ ;

$N_{ijk} = \langle ax^i, bx^j, dx^k \rangle$  (or  $\langle ax^i, bx^j, dx^k \rangle$  for (J6)), where  $0 \leq i, j, k \leq p-1$ .

It is easy to see that  $|N'_i| = p$ . By Lemma 2.6,  $\alpha_1(N_i) = p^2$ . Since  $N_{ijk} \cong K$ ,  $\alpha_1(N_{ijk}) = p + p^2$ . Hence

$$\alpha_1(G) = \frac{1}{1+p} \sum_{H \in \Gamma_1} \alpha_1(H) = \frac{1}{1+p} ((p+p^2)p^2 + p^3(p+p^2)) = p^3 + p^4.$$

□

## 5.2 $G$ has no abelian subgroup of index $p$

In this section assume  $G$  is an  $\mathcal{A}_3$ -group without an abelian subgroup of index  $p$  and an  $\mathcal{A}_1$ -subgroup of index  $p$  in Theorem 5.6, 5.7, 5.13, 5.15 and 5.16.

**Theorem 5.6.** *Suppose that  $G$  is a  $p$ -group all of whose maximal subgroups are  $\mathcal{A}_2$ -groups, and every maximal subgroup of  $G$  is generated by two elements. Then  $G$  is an  $\mathcal{A}_3$ -group if and only if  $G$  is isomorphic to one of the following pairwise non-isomorphic groups:*

(Ki)  $G$  is metacyclic. In this case,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1+p)$  and  $\alpha_1(G) = 1 + p + p^2$ .

(K1)  $\langle a, b \mid a^{p^{r+3}} = 1, b^{p^{r+s+t}} = a^{p^{r+s}}, [a, b] = a^{p^r} \rangle$ , where  $p > 2$  and  $r, s, t$  are non-negative integers with  $r \geq 1$  and  $r + s \geq 3$ ;  $|G| = p^{2r+s+t+3}$ ,  $c(G) = 4$  for  $r = 1$ ;  $c(G) = 3$  for  $r = 2$  and  $c(G) = 2$  for  $r > 2$ ,  $\Phi(G) = \langle a^p, b^p \rangle \cong M_p(r+2, r+s+t-1)$  if  $s \geq 3$ ,  $\Phi(G) = \langle a^p, b^p \rangle \cong M_p(r+t+2, r+s-1)$  if  $s < 3$ ; moreover,  $G' = \langle a^{p^r} \rangle \cong C_{p^3}$ ,  $Z(G) = \langle a^{p^3}, b^{p^3} \rangle \cong C_{p^r} \times C_{p^{r+t+s-3}}$  if  $s \geq 3$ ,  $Z(G) = \langle a^{p^3}, b^{p^3} \rangle \cong C_{p^{r+s-3}} \times C_{p^{r+t}}$  if  $s < 3$ .

(K2)  $\langle a, b \mid a^{2^{r+3}} = 1, b^{2^{r+s+t}} = a^{2^{r+s}}, [a, b] = a^{2^r} \rangle$ , where  $r, s, t$  are non-negative integers with  $r \geq 2$  and  $r + s \geq 3$ . Moreover,  $|G| = 2^{2r+s+t+3}$ ,  $c(G) = 3$  for  $r = 2$  and  $c(G) = 2$  for  $r > 2$ ,  $\Phi(G) = \langle a^2, b^2 \rangle \cong M_2(r+2, r+s+t-1)$  if  $s \geq 3$ ,  $\Phi(G) = \langle a^2, b^2 \rangle \cong M_2(r+t+2, r+s-1)$  if  $s < 3$ ;  $G' = \langle a^{2^r} \rangle \cong C_{2^3}$ ,  $Z(G) = \langle a^8, b^8 \rangle \cong C_{2^r} \times C_{2^{r+t+s-3}}$  if  $s \geq 3$ ,  $Z(G) = \langle a^{2^3}, b^{2^3} \rangle \cong C_{2^{r+s-3}} \times C_{2^{r+t}}$  if  $s < 3$ .

(Kii)  $G$  is not metacyclic. In this case,  $p > 2$ ,  $|G| = p^6$ ,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1+p)$  and  $\alpha_1(G) = p + p^2$ .

(K3)  $\langle a, b, c \mid a^{p^2} = b^{p^2} = c^{p^2} = 1, [a, b] = c, [c, b] = a^p c^{mp}, [c, a] = b^{\nu p} c^{np}, [a, b^p] = [a^p, b] = c^p, [c, a^p] = [c, b^p] = [c^p, a] = [c^p, b] = 1 \rangle$ , where  $\nu$  is a fixed quadratic non-residue modulo  $p$ . The parameters  $m, n$  are the smallest positive integers satisfying  $(m-1)^2 - \nu^{-1}(n+\nu)^2 \equiv r \pmod{p}$ , for  $r = 0, 1, \dots, p-1$ ; moreover,  $|G| = p^6$ ,  $c(G) = 4$ ,  $\Phi(G) = G' = \langle a^p, b^p, c \rangle \cong C_p \times C_p \times C_{p^2}$  and  $Z(G) = \langle c^p \rangle \cong C_p$ .



- (K4)  $\langle a, b; c, d \mid a^9 = b^9 = c^3 = d^3 = 1, [a, b] = c, [c, b] = a^3, [c, a] = b^{-3}, [a^3, b] = [a, b^3] = d, [d, a] = [d, b] = 1 \rangle$ ; where  $|G| = 3^6$ ,  $c(G) = 4$ ,  $\Phi(G) = G' = \langle a^3, b^3, c, d \rangle \cong C_3^4$  and  $Z(G) = \langle d \rangle \cong C_3$ .
- (K5)  $\langle a, b; c, d \mid a^9 = b^9 = c^3 = d^3 = 1, [a, b] = c, [c, b] = a^3d, [c, a] = b^{-3}d, [a^3, b] = [a, b^3] = d, [d, a] = [d, b] = 1 \rangle$ , where  $|G| = 3^6$ ,  $c(G) = 4$ ,  $\Phi(G) = G' = \langle a^3, b^3, c, d \rangle \cong C_3^4$  and  $Z(G) = \langle d \rangle \cong C_3$ .

**Proof** Since all non-abelian proper subgroups of  $G$  are generated by two elements,  $G$  is one of the groups classified in [29]. It follows from [29, Main Theorem],  $G$  is either a metacyclic group or a 3-group of maximal class or the group in [29, Theorem 5.5 & 5.6].

If  $G$  is metacyclic, then  $|G'| = p^3$  by Lemma 3.1. By using the classification of metacyclic  $p$ -groups in [28, 30] and checking their maximal subgroups, we get the groups (K1)–(K2). The details is omitted.

If  $G$  is of maximal class, then, by Lemma 2.4,  $G$  does not satisfy the hypothesis.

If  $G$  is the group in [29, Theorem 5.5], then  $G$  is the group (K3). If  $G$  is the group in [29, Theorem 5.6], then  $G$  is the group (K4)–(K5).

Since  $d(G) = 2$ ,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1 + p)$ . If  $G$  is one of the groups (K1)–(K2), the  $G$  is metacyclic and  $\Phi(G) \in \mathcal{A}_1$ . Let  $H \in \Gamma_1$ . Then  $H$  is also metacyclic and  $\alpha_1(H) = 1 + p$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) - p \sum_{H \in \Gamma_2} \alpha_1(H) = \mu_2(1 + p) - p = p^2 + p + 1.$$

If  $G$  is one of the groups (K3)–(K5), then  $\Phi(G)$  is abelian. Let  $H \in \Gamma_1$ . Then  $\alpha_1(H) = p$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = \mu_2 p = p^2 + p.$$

□

**Theorem 5.7.** Suppose that  $|G| = p^n$  and  $G$  is an  $\mathcal{A}_3$ -group. If  $G$  has a three-generator maximal subgroup  $M$  such that  $d(M) = 3$  and  $M' \not\leq Z(G)$ , then  $p \geq 5$ ,  $n = 6$ , all maximal subgroup of  $G$  are  $\mathcal{A}_2$ -groups and  $G$  is one of the following non-isomorphic groups:

- (L1)  $\langle x, m; a \mid x^{p^2} = m^{p^2} = a^{p^2} = 1, [x, m] = a, [a, x] = x^p, [a, m] = m^{-p} \rangle$ ; where  $|G| = p^6$ ,  $c(G) = 4$ ,  $\Phi(G) = G' = \langle a, x^p, m^p \rangle \cong C_p^2 \times C_{p^2}$  and  $Z(G) = \langle a^p \rangle \cong C_p$ .
- (L2)  $\langle x, m; a \mid x^{p^2} = m^{p^2} = a^{p^2} = 1, [x, m] = a, [a, x] = x^p a^p, [a, m] = m^{-p} a^{vp} \rangle$ , where  $v \in F_p$ . Moreover,  $|G| = p^6$ ,  $c(G) = 4$ ,  $\Phi(G) = G' = \langle a, x^p, m^p \rangle \cong C_p^2 \times C_{p^2}$  and  $Z(G) = \langle a^p \rangle \cong C_p$ .

Moreover,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1 + p)$  and  $\alpha_1(G) = 3p^2 + p$ .

**Proof** By Lemma 3.16, we may assume  $|M'| \leq p^2$ . If  $|M'| = p$ , then  $M' \leq Z(G)$ . Hence  $|M'| = p^2$ . It follows from Lemma 2.6(4) that  $c(M) = 2$ ,  $M' \cong C_p^2$ ,  $\Phi(M) = \mathcal{U}_1(M) = Z(M)$ ,  $M$  has a unique abelian subgroup  $A$  of index  $p$  and  $A/M'$  has type  $(p^{n-6}, p, p)$ . Since  $A$  is characteristic in  $M$  and  $M \trianglelefteq G$ , we have  $A \trianglelefteq G$ . We take  $x \in G \setminus M$  and let  $H = \langle x, M' \rangle$ . Then  $H$  is not abelian since  $M' \not\leq Z(G)$ . Moreover,  $|H'| = p$ ,  $H' \leq M'$  and  $H' \leq Z(G)$ . By Lemma 2.2 we have  $H$  is an  $\mathcal{A}_1$ -group. It follows that  $|H| \geq p^{n-2}$ . Hence  $o(xM') \geq p^{n-4}$ . We will prove that

$$(1) \Phi(G) \leq A.$$

Otherwise,  $G/A$  is cyclic,  $G = \langle x, A \rangle$  and  $M = \langle x^p, A \rangle$ . Since  $x^{p^2} \in \mathcal{U}_1(M) = Z(M)$ , we have  $x^{p^2} \in Z(G)$ . It follows that

$$|Z(G)/H'| = |Z(G)/Z(G) \cap M'| = |Z(G)M'/M'| \geq o(x^{p^2}M') \geq p^{n-6}.$$

Hence  $|Z(G)| \geq p^{n-5}$ . By Lemma 2.8,  $|G'| = |A/Z(G)| \leq p^3$ . Then  $G_3 \leq M'$ ,  $c(G) \leq 4$  and  $G_4 \leq Z(G)$ . Since  $G' \geq M' \cong C_p^2$ ,  $d(G') \geq 2$  and hence  $|\Phi(G')| \leq p$ . It follows that  $\Phi(G') \leq H'$ .

We claim that  $[x, w, x]^{\binom{p}{2}} \in Z(G)$  for all  $w \in G$ . Otherwise, we may assume that  $p = 2$  since  $[x, w, x]^{\binom{p}{2}} \in \Phi(G') \leq Z(G)$  for  $p > 2$ . In this case,  $[x, w, x, x] \neq 1$ . Let  $L = \langle [x, w], x \rangle$ . Then  $c(L) = 3$ . Since  $G$  is an  $\mathcal{A}_3$ -group,  $L$  is an  $\mathcal{A}_2$ -group. By Lemma 2.6 (6),  $L' \cong C_4$ . On the other hand,  $L' \leq G_3 \leq M'$ , a contradiction.

Since  $[x^p, w] \equiv [x, w]^p[x, w, x]^{\binom{p}{2}} \pmod{G_4}$  and  $\Phi(G') \leq Z(G)$ ,  $[x^p, w] \in Z(G)$ . By Lemma 2.8,  $M' = \{[x^p, a] \mid a \in A\} \leq Z(G)$ , a contradiction.

$$(2) |G| = p^6, o(xM') = p^2 \text{ and } x^p \in A \setminus M'.$$

By (1),  $x^p \in A$ . Since  $o(xM') \geq p^{n-4}$ , we have  $o(x^pM') \geq p^{n-5}$  and hence  $\exp(A/M') \geq p^{n-5}$ . On the other hand,  $A/M'$  has type  $(p^{n-6}, p, p)$ . If  $n \geq 7$ , then  $\exp(A/M') = p^{n-6}$ , a contradiction. Hence  $n = 6$ ,  $\exp(A/M') = p$ ,  $o(xM') = p^2$  and  $x^p \in A \setminus M'$ .

(3)  $G = \langle x, m; a \mid x^{p^2} = m^{p^2} = a^{p^2} = 1, [x, m] = a, [a, x] = x^p a^{up}, [a, m] = m^{-p} a^{vp} \rangle$ , where  $u, v \in F_p$ .

Let  $A = \langle x^p, a, M' \rangle$  and  $M = \langle m, A \rangle$ . Then  $M' = \langle [m, x^p] \rangle \times \langle [m, a] \rangle$ . Since  $[x^p, x, m] = 1$  and  $[x, m, x^p] = 1$ , we have  $[m, x^p, x] = 1$  by Witt's formula. Hence  $Z = \langle [m, x^p] \rangle$ . Since  $M' \not\leq Z(G)$ ,  $[m, a, x] \neq 1$ . Let  $[m, a, x] = [m, x^p]^i$ , where  $(i, p) = 1$ . By calculation we have

$$[[a, x]x^{-ip}, m] = [a, x, m][x^{-ip}, m] = [a, x, m][m, a, x] = 1.$$

It follows that  $[a, x]x^{-ip} \in Z(M) = M'$ .

We claim that  $x^{p^2} = 1$ . Otherwise,  $[a, x]^p \neq 1$ . Hence  $\langle a, x \rangle \in \mathcal{A}_2$ . By Lemma 2.6 (7),  $\langle a, x \rangle$  is metacyclic. If  $p > 2$ , then  $[a, x]^p = [a, x] \neq 1$ , a contradiction. If

$p = 2$ , then, by calculation, we have  $1 = [a, x^2] = [a, x]^2[a, x, x]$ . Hence  $[a, x, x] = x^4$ . By calculation,  $[a^2, x] = [a, x]^2[a, x, a] = x^4$ . Hence  $a^2 \notin Z(G)$ . It follows that  $M' = \langle a^2, x^4 \rangle$ . Since  $[a, x, x] \neq 1$ , we have  $[a, x] \equiv x^2 a^2 \pmod{Z(G)}$ . By calculation we have  $\langle a^2, ax \rangle \cong D_8$ , which is contrary to that  $G$  is an  $\mathcal{A}_3$ -group. Hence  $x^{p^2} = 1$ .

By Lemma 2.6 (4),  $\exp(A) > p$ . It follows that  $a^p \neq 1$ . Since  $[a^p, x] = [a, x]^p = 1$ ,  $Z(G) = \langle a^p \rangle$  and hence  $M' = \langle a^p, [a, m] \rangle$ .

We claim  $p > 2$ . Otherwise, since  $[a, x^2] = 1$ , we have  $[a, x, x] = 1$ . Hence we may assume  $[a, x] = x^2 a^{2u}$ . By calculation we have  $\langle [a, m], ax \rangle \cong D_8$ , which is contrary to that  $G$  is an  $\mathcal{A}_3$ -group. Hence  $p \geq 3$ .

Since  $[a, x, x] \in H'$ , we may assume  $[a, x, x] = a^{sp}$ . Since  $1 \neq [a, m, x] \in H'$ , we may assume  $[a, m, x] = a^{tp}$  where  $(t, p) = 1$ . Let  $j = -2^{-1}t^{-1}s$ . Then

$$[a, m^j x, m^j x] = [a, x, x][a, x, m]^j [a, m, x]^j = a^{(s+2jtp)} = 1.$$

By replacing  $x$  with  $xm^j$  we have  $[a, x, x] = 1$ .

Since  $[x^p, m] \neq 1$ , we have  $[x, m]^p \neq 1$  and  $[x, m^p] \neq 1$ . Hence  $M' = \langle a^p, m^p \rangle$  and we may assume  $[x, m] \equiv a^v \pmod{\langle x^p, M' \rangle}$ , where  $(v, p) = 1$ . By replacing  $m$  with  $m^{v^{-1}}$  we have  $[x, m] \equiv a \pmod{\langle x^p, M' \rangle}$ . Without loss of generality assume that  $[x, m] = a$ . Hence  $G = \langle x, m \rangle$  satisfy the following relations:

$$x^{p^2} = m^{p^2} = a^{p^2} = 1, [x, m] = a, [a, x] \equiv x^{ip} \pmod{\langle a^p \rangle}, [a, m] \equiv m^{jp} \pmod{\langle a^p \rangle}.$$

By replacing  $m$  and  $a$  with  $m^{i^{-1}}$  and  $[x, m^{i^{-1}}]$  respectively, we have  $[a, x] \equiv x^p \pmod{\langle a^p \rangle}$ . Since  $G$  is metabelian, we have  $[a, x, m] = [a, m, x]$ . It follows that  $j = -1$ . If  $p = 3$ , then  $\langle x^3, xm \rangle$  is not abelian and is of order  $3^3$ , which is contrary to that  $G$  is an  $\mathcal{A}_3$ -group. Hence  $p \geq 5$  and we may assume that

$$G = \langle x, m; a \mid x^{p^2} = m^{p^2} = a^{p^2} = 1, [x, m] = a, [a, x] = x^p a^{up}, [a, m] = m^{-p} a^{vp} \rangle,$$

where  $u, v \in F_p$ .

(4) Two groups with parameters  $(u, v)$  and  $(u', v')$  are pairwise isomorphic if and only if there exists  $t \in F_p^*$  such that  $(u', v') = (u, v) \begin{bmatrix} t \\ t^{-1} \end{bmatrix}$  or  $(u', v') = (v, u) \begin{bmatrix} t \\ t^{-1} \end{bmatrix}$ .

Suppose that  $G = \langle x, m, a \rangle$  and  $\bar{G} = \langle \bar{x}, \bar{m}, \bar{a} \rangle$  with parameters  $(u, v)$  and  $(u', v')$  respectively. Let  $\theta$  be an isomorphism from  $\bar{G}$  to  $G$ . Assume that

$$\bar{x}^\theta = x^{s_{11}} m^{s_{12}} a^{s_{13}} r_1, \bar{m}^\theta = x^{s_{21}} m^{s_{22}} a^{s_{23}} r_2,$$

where  $s_{ij} \in F_p$  such that  $s := s_{11}s_{22} - s_{12}s_{21} \in F_p^*$  and  $r_1, r_2 \in G_3$ .

By calculation,  $\bar{a}^\theta = [\bar{x}, \bar{m}]^\theta \equiv a^s \pmod{G_3}$ . Since  $[\bar{a}^\theta, \bar{x}^\theta] \equiv (\bar{x}^p)^\theta \equiv (\bar{x}^p)^p \pmod{G_4}$ , we have  $[a^s, x^{s_{11}} m^{s_{12}}] \equiv (x^{s_{11}} m^{s_{12}})^p \pmod{G_4}$ . Comparing indexes of  $x^p$  and  $m^p$  in two sides, we have  $ss_{11} = s_{11}$  and  $ss_{12} = -s_{12}$ . Since  $[\bar{a}^\theta, \bar{m}^\theta] = (\bar{m}^{-p})^\theta = (\bar{m}^\theta)^{-p}$ , we

have  $[a^s, x^{s_{21}}m^{s_{22}}] \equiv (x^{s_{21}}m^{s_{22}})^{-p} \pmod{G_4}$ . Comparing indexes of  $x^p$  and  $m^p$  in two sides, we have  $ss_{21} = -s_{21}$  and  $ss_{22} = s_{22}$ . It follows that

$$s_{11}s_{22} = 1, s_{12} = s_{21} = 0 \text{ or } s_{11} = s_{22} = 0, s_{12}s_{21} = 1.$$

If  $s_{11}s_{22} = 1$  and  $s_{12} = s_{21} = 0$ , then, by calculation,

$$\bar{a}^\theta = [\bar{x}, \bar{m}]^\theta = [x^{s_{11}}a^{s_{13}}, m^{s_{22}}a^{s_{23}}] \equiv ax^{-s_{23}s_{11}p}m^{-s_{13}s_{22}p} \pmod{G_4}.$$

Since

$$[\bar{a}^\theta, \bar{x}^\theta] = (\bar{x}^p\bar{a}^{u'p})^\theta = (\bar{x}^\theta)^p(\bar{a}^\theta)^{u'p},$$

we have

$$[ax^{-s_{23}s_{11}p}m^{-s_{13}s_{22}p}, x^{s_{11}}] = (x^{s_{11}}a^{s_{13}})^pa^{u'p}.$$

Comparing index of  $a^p$  in two sides, we have

$$s_{11}u = u'. \quad (1)$$

Since

$$[\bar{a}^\theta, \bar{m}^\theta] = (\bar{m}^{-p}\bar{a}^{v'p})^\theta = (\bar{m}^\theta)^{-p}(\bar{a}^\theta)^{v'p},$$

we have

$$[ax^{-s_{23}s_{11}p}, m^{s_{22}}] = (m^{s_{22}}a^{s_{23}})^{-p}a^{v'p}.$$

Comparing index of  $a^p$  in two sides, we have

$$s_{22}v = v'. \quad (2)$$

Let  $t = s_{11}$ . Then  $(u', v') = (u, v) \begin{bmatrix} t \\ t^{-1} \end{bmatrix}$ .

On the other hand, if there exists  $t \in F_p^*$  such that  $(u', v') = (u, v) \begin{bmatrix} t \\ t^{-1} \end{bmatrix}$ , then  $\theta : \bar{x} \mapsto x^t, \bar{m} \mapsto m^{t^{-1}}$  is an isomorphism from  $\bar{G}$  to  $G$ .

If  $s_{11} = s_{22} = 0$  and  $s_{12}s_{21} = 1$ , then, by calculation,

$$\bar{a}^\theta = [\bar{x}, \bar{m}]^\theta = [m^{s_{12}}a^{s_{13}}, x^{s_{21}}a^{s_{23}}] \equiv a^{-1}x^{s_{13}s_{21}p}m^{s_{12}s_{23}p} \pmod{G_4}.$$

Since

$$[\bar{a}^\theta, \bar{x}^\theta] = (\bar{x}^p\bar{a}^{u'p})^\theta = (\bar{x}^\theta)^p(\bar{a}^\theta)^{u'p},$$

we deduce that

$$[a^{-1}x^{s_{13}s_{21}p}m^{s_{12}s_{23}p}, m^{s_{12}}] = (m^{s_{12}}a^{s_{13}})^pa^{u'p}.$$

Comparing index of  $a^p$  in two sides, we have

$$-s_{12}v = u'. \quad (3)$$

Since

$$[\bar{a}^\theta, \bar{m}^\theta] = (\bar{m}^{-p} \bar{a}^{v'p})^\theta = (\bar{m}^\theta)^{-p} (\bar{a}^\theta)^{v'p},$$

we deduce that

$$[a^{-1} x^{s_{13} s_{21} p} m^{s_{12} s_{23} p}, x^{s_{21}}] = (x^{s_{21}} a^{s_{23}})^{-p} a^{v'p}.$$

Comparing index of  $a^p$  in two sides, we have

$$-s_{21}u = v'. \quad (4)$$

Let  $t = -s_{12}$ . Then  $(u', v') = (v, u) \begin{bmatrix} t \\ t^{-1} \end{bmatrix}$ .

On the other hand, if there exists  $t \in F_p^*$  such that  $(u', v') = (v, u) \begin{bmatrix} t \\ t^{-1} \end{bmatrix}$ , then  $\theta : \bar{x} \mapsto m^{-t}, \bar{m} \mapsto x^{-t^{-1}}$  is an isomorphism from  $\bar{G}$  to  $G$ .

(5)  $G$  is one of the groups in the theorem. That is, we may assume that  $u = v = 0$  or  $u = 1$ .

If  $u = v = 0$ , then  $G$  is the group (L1). If  $u \neq 0$ , then, replacing  $x$  and  $m$  with  $x^{u^{-1}}$  and  $m^u$  respectively,  $G$  is the group (L2). If  $v \neq 0$ , then, replacing  $x$  and  $m$  with  $m^{v^{-1}}$  and  $x^v$  respectively,  $G$  is the group (L2).

By (4), groups in Theorem 5.7 are pairwise non-isomorphic.

Now we calculate  $(\mu_0, \mu_1, \mu_2)$  and  $\alpha_1(G)$ .

Since  $d(G) = 2$ ,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1 + p)$ . All maximal subgroups of  $G$  are:

$$M = \langle a, x, \Phi(G) \rangle;$$

$$M_i = \langle a, mx^i, \Phi(G) \rangle, \text{ where } 0 \leq i \leq p-1.$$

It is easy to see that  $M = \langle a, x, m^p \rangle$  such that  $|M'| = p^2$  and  $d(M) = 3$ . Hence  $\alpha_1(M) = p^2 + p$  and  $\Phi(G)$  is the unique abelian maximal subgroup of  $M$ . Similarly,  $M_0 = \langle a, m, x^p \rangle$  and  $\alpha(M_0) = p^2 + p$ . By calculation,  $d(M_i) = 2$  for  $i \neq 0$ . Hence  $\alpha_1(M_i) = p$  for  $i \neq 0$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = 2 \times (p + p^2) + (p-1) \times p = 3p^2 + p.$$

□

**Lemma 5.8.** Suppose that  $G(i, j) = \langle b, a_1; a_2, a_3 \mid b^{p^2} = a_1^p = a_2^p = a_3^p = 1, [a_1, b] = a_2, [a_2, b] = a_3, [a_3, b] = b^{ip}, [a_2, a_1] = b^{jp}, [a_3, a_1] = 1 \rangle$ , where  $i, j \in F_p^*$  and  $p \geq 5$ .

- (1) If  $G = G(i, j)$ , then  $|G| = p^5$ ,  $G' = \langle a_2, a_3, b^p \rangle$ ,  $G_3 = \langle a_3, b^p \rangle$ ,  $G_4 = \langle b^p \rangle$ ,  $G \in \mathcal{A}_3$ , and  $M = \langle b^p, a_1, a_2, a_3 \rangle$  is the unique three-generator maximal subgroup of  $G$ ;
- (2)  $G(i, j) \cong G(i', j')$  if and only if there exist  $r, s \in F_p^*$  such that  $j' = s^2 j$  and  $i' = r^2 s i$ .

**Proof** (1) Let

$$K = \langle a_2, a_1; d \mid a_2^p = a_1^p = d^p = 1, [a_2, a_1] = d^j, [d, a_1] = [d, a_2] = 1 \rangle \times \langle a_3 \rangle.$$

We define an automorphism  $\beta$  of  $K$  as follows:

$$a_2^\beta = a_2 a_3, \quad a_1^\beta = a_1 a_2 \quad \text{and} \quad a_3^\beta = a_3 d^i.$$

Then  $o(\beta) = p$ . It follows from the cyclic extension theory that  $G = \langle K, b \rangle$  is a cyclic extension of  $K$  by  $C_p$ . Hence  $|G| = p^5$ . It is easy to verify that  $G' = \langle a_2, a_3, b^p \rangle$ ,  $G_3 = \langle a_3, b^p \rangle$ ,  $G_4 = \langle b^p \rangle$ ,  $G \in \mathcal{A}_3$ , and  $M = \langle b^p, a_1, a_2, a_3 \rangle$  is the unique three-generator maximal subgroup of  $G$ .

(2) For convenience, let  $G = \langle b, a_1, a_2, a_3 \rangle \cong G(i, j)$ ,  $\bar{G} = \langle \bar{b}, \bar{a}_1, \bar{a}_2, \bar{a}_3 \rangle \cong G(i', j')$  and  $\theta$  be an isomorphism from  $\bar{G}$  to  $G$ . Since  $M = \langle b^p, a_1, a_2, a_3 \rangle$  and  $\bar{M} = \langle \bar{b}^p, \bar{a}_1, \bar{a}_2, \bar{a}_3 \rangle$  are the unique three-generator maximal subgroups of  $G$  and  $\bar{G}$  respectively, we have  $\bar{M}^\theta = M$ . Hence we may assume that  $\bar{b}^\theta = b^r x$  and  $\bar{a}_1^\theta = a_1^s y$ , where  $r, s \in F_p^*$  and  $x \in M, y \in \Phi(G)$ .

By calculation we have  $\bar{a}_2^\theta = [\bar{a}_1, \bar{b}]^\theta \equiv a_2^{rs} \pmod{G_3}$ ,  $\bar{a}_3^\theta = [\bar{a}_2, \bar{b}]^\theta \equiv a_3^{r^2 s} \pmod{G_4}$ .

Since  $[\bar{a}_2^\theta, \bar{a}_1^\theta] = (\bar{b}^{j'p})^\theta = (\bar{b}^\theta)^{j'p}$ , we have  $[a_2^{rs}, a_1^s] = b^{rj'p}$ . Left side of the equation is  $b^{rs^2jp}$ . By comparing index of  $b^p$  in two sides we have  $j' = s^2 j$ .

Since  $[\bar{a}_3^\theta, \bar{b}^\theta] = (\bar{b}^{i'p})^\theta = (\bar{b}^\theta)^{i'p}$ , we have  $[a_3^{r^2 s}, b^r] = b^{ri'p}$ . Left side of the equation is  $b^{r^3 sip}$ . By comparing index of  $b^p$  in two sides we have  $i' = r^2 si$ .

On the other hand, if there exist  $r, s \in F_p^*$  such that  $j' = s^2 j$  and  $i' = r^2 si$ , then,  $\theta : \bar{a}_1 \rightarrow a_1^s, \bar{b} \rightarrow b^r$  is an isomorphism from  $\bar{G}$  to  $G$ .  $\square$

**Lemma 5.9.** Suppose that  $G(i, j) = \langle b, a_1; a_2, a_3 \mid b^p = a_1^{p^2} = a_2^p = a_3^p = 1, [a_1, b] = a_2, [a_2, b] = a_3, [a_3, b] = a_1^{ip}, [a_2, a_1] = a_1^{jp}, [a_3, a_1] = 1 \rangle$ , where  $i, j \in F_p^*$  and  $p \geq 5$ .

- (1) If  $G = G(i, j)$ , then  $|G| = p^5$ ,  $G' = \langle a_2, a_3, a_1^p \rangle$ ,  $G_3 = \langle a_3, a_1^p \rangle$ ,  $G_4 = \langle a_1^p \rangle$ ,  $G \in \mathcal{A}_3$ , and  $M = \langle a_1, a_2, a_3 \rangle$  is the unique three-generator maximal subgroup of  $G$ ;
- (2)  $G(i, j) \cong G(i', j')$  if and only if there exist  $r, s \in F_p^*$  such that  $j' = rsj$  and  $i' = r^3 i$ .

**Proof** (1) Let

$$K = \langle a_2, a_1 \mid a_2^p = a_1^{p^2} = 1, [a_2, a_1] = a_1^{ip} \rangle \times \langle a_3 \rangle.$$

We define an automorphism  $\beta$  of  $K$  as follows:

$$a_2^\beta = a_2 a_3, \quad a_1^\beta = a_1 a_2 \quad \text{and} \quad a_3^\beta = a_3 a_1^{ip}.$$

Then  $o(\beta) = p$ . Hence  $G = K \rtimes \langle b \rangle$  and  $|G| = p^5$ . It is easy to verify that  $G' = \langle a_2, a_3, a_1^p \rangle$ ,  $G_3 = \langle a_3, a_1^p \rangle$ ,  $G_4 = \langle a_1^p \rangle$ ,  $G \in \mathcal{A}_3$ , and  $M = \langle a_1, a_2, a_3 \rangle$  is the unique three-generator maximal subgroup of  $G$ .

(2) For convenience, let  $G = \langle b, a_1, a_2, a_3 \rangle \cong G(i, j)$ ,  $\bar{G} = \langle \bar{b}, \bar{a}_1, \bar{a}_2, \bar{a}_3 \rangle \cong G(i', j')$  and  $\theta$  be an isomorphism from  $\bar{G}$  to  $G$ . Since  $M = \langle a_1, a_2, a_3 \rangle$  and  $\bar{M} = \langle \bar{a}_1, \bar{a}_2, \bar{a}_3 \rangle$  are the unique three-generator maximal subgroups of  $G$  and  $\bar{G}$  respectively, we have  $\bar{M}^\theta = M$ . Hence we may assume that  $\bar{b}^\theta = b^r x$ ,  $\bar{a}_1^\theta = a_1^s y$ , where  $r, s \in F_p^*$  and  $x \in \Omega_1(M)$ ,  $y \in \Phi(G)$ .

By calculation we have  $\bar{a}_2^\theta = [\bar{a}_1, \bar{b}]^\theta \equiv a_2^{rs} \pmod{G_3}$ .  $\bar{a}_3^\theta = [\bar{a}_2, \bar{b}]^\theta \equiv a_3^{r^2s} \pmod{G_4}$ .

Since  $[\bar{a}_2^\theta, \bar{a}_1^\theta] = (\bar{a}_1^{j'p})^\theta = (\bar{a}_1^\theta)^{j'p}$ , we have  $[a_2^{rs}, a_1^s] = a_1^{sj'p}$ . Left side of the equation is  $a_1^{rs^2jp}$ . By comparing index of  $a_1^p$  in two sides we have  $j' = rsj$ .

Since  $[\bar{a}_3^\theta, \bar{b}^\theta] = (\bar{a}_1^{i'p})^\theta = (\bar{a}_1^\theta)^{i'p}$ , we have  $[a_3^{r^2s}, b^r] = a_1^{si'p}$ . Left side of the equation is  $a_1^{r^3sip}$ . By comparing index of  $a_1^p$  in two sides we have  $i' = r^3i$ .

On the other hand, if there exist  $r, s \in F_p^*$  such that  $j' = rsj$  and  $i' = r^3i$ , then  $\theta : \bar{a}_1 \rightarrow a_1^s, \bar{b} \rightarrow b^r$  is an isomorphism from  $\bar{G}$  to  $G$ .  $\square$

**Lemma 5.10.** *Suppose that  $p \geq 3$  and  $G = G(r, s) = \langle b, a_1; a_2 \mid b^{p^3} = a_1^{p^2} = a_2^{p^2} = 1, [a_1, b] = a_2, [a_2, a_1] = b^{\nu_1 p^2} a_2^{rp}, [a_2, b] = a_1^{\nu_2 p} a_2^{sp} \rangle$ , where  $r, s \in F_p$  and  $\nu_1, \nu_2 = 1$  or a fixed quadratic non-residue modula  $p$ . Then*

(1)  $|G| = p^7$ ,  $G' = \langle a_1^p, a_2, b^{p^2} \rangle$ ,  $G_3 = \langle a_1^p, a_2^p, b^{p^2} \rangle$ ,  $G_4 = \langle a_2^p \rangle$ ,  $Z(G) \cap G' = \langle a_2^p, b^{p^2} \rangle$  and  $M = \langle b^p, a_1, a_2 \rangle$  is the unique three-generator maximal subgroup of  $G$ ;

(2)  $G$  is one of the following non-isomorphic groups:

- (i)  $\langle b, a_1; a_2 \mid b^{p^3} = a_1^{p^2} = a_2^{p^2} = 1, [a_1, b] = a_2, [a_2, a_1] = b^{\nu_1 p^2}, [a_2, b] = a_1^{\nu_2 p} a_2^{sp} \rangle$ , where  $\nu_1, \nu_2 = 1$  or a fixed quadratic non-residue modula  $p$  and  $s = \nu_2, \nu_2 + 1, \dots, \nu_2 + \frac{p-1}{2}$ ;
- (ii)  $\langle b, a_1; a_2 \mid b^{p^3} = a_1^{p^2} = a_2^{p^2} = 1, [a_1, b] = a_2, [a_2, a_1] = b^{\nu_1 p^2} a_2^{rp}, [a_2, b] = a_1^{\nu_2 p} \rangle$ , where  $\nu_1, \nu_2 = 1$  or a fixed quadratic non-residue modula  $p$  and  $r = 1, 2, \dots, \frac{p-1}{2}$ .

(3)  $[a_1^p, b] = a_2^p$ ;

(4) all maximal subgroups of  $G$ , except  $M$ , are  $\mathcal{A}_2$ -groups;

(5) If  $p = 3$  and  $\nu_2 = -1$ , then  $|M'| = 3$  and  $G \in \mathcal{A}_3$ ;

(6) If  $p = 3$  and  $\nu_2 = 1$ , then  $|M'| = 9$ , and  $G \in \mathcal{A}_3$  if and only if  $r^2 + 4\nu_1$  is not a square;

(7) If  $p \geq 5$ , then  $|M'| = p^2$ , and  $G \in \mathcal{A}_3$  if and only if  $r^2 - 4\nu_1$  is not a square.

**Proof** (1) Let

$$K = \langle a_2, a_1; d \mid a_2^{p^2} = a_1^{p^2} = d^p = 1, [a_2, a_1] = d \rangle.$$

We define an automorphism  $\beta$  of  $K$  as follows:

$$a_2^\beta = a_2 a_1^{\nu_2 p} a_2^{sp}, \quad a_1^\beta = a_1 a_2.$$

Then

$$a_2^{\beta p} = a_2, a_1^{\beta p^2} = 1, o(\beta) \leq p^2.$$

It follows from the cyclic extension theory that  $G = \langle K, b \rangle$  is a cyclic extension of  $K$  by  $C_{p^2}$ . Hence  $|G| = p^7$ . It is easy to check that  $G' = \langle a_1^p, a_2, b^{p^2} \rangle$ ,  $G_3 = \langle a_1^p, a_2^p, b^{p^2} \rangle$ ,  $G_4 = \langle a_2^p \rangle$ ,  $Z(G) \cap G' = \langle a_2^p, b^{p^2} \rangle$  and  $M = \langle b^p, a_1, a_2 \rangle$  is the unique three-generator maximal subgroup of  $G$ .

(2) For convenience, let  $G = \langle b, a_1, a_2 \rangle \cong G(r, s)$ ,  $\bar{G} = \langle \bar{b}, \bar{a}_1, \bar{a}_2 \rangle \cong G(r', s')$  and  $\theta$  be an isomorphism from  $\bar{G}$  to  $G$ . Since  $M = \langle b^p, a_1, a_2 \rangle$  and  $\bar{M} = \langle \bar{b}^p, \bar{a}_1, \bar{a}_2 \rangle$  are the unique three-generator maximal subgroups of  $G$  and  $\bar{G}$  respectively, we have  $\bar{M}^\theta = M$ . Hence we may assume that  $\bar{b}^\theta = b^l a_1^m x$ ,  $\bar{a}_1^\theta = a_1^i a_2^j b^{kp} y$ , where  $l, i \in F_p^*$  and  $x \in \Phi(G)$ ,  $y \in \bar{U}_1(M)$ .

By calculation,  $\bar{a}_2^\theta = [\bar{a}_1, \bar{b}]^\theta \equiv a_2^{il} \pmod{G_3}$ .

Since  $[\bar{a}_2^\theta, \bar{a}_1^\theta] = (\bar{b}^{\nu_1 p^2} \bar{a}_2^{r' p})^\theta = (\bar{b}^\theta)^{\nu_1 p^2} (\bar{a}_2^\theta)^{r' p}$ , we have  $[a_2^{il}, a_1^i] = b^{\nu_1 p^2} a_2^{il r' p}$ . Left side of above equation is  $b^{i^2 \nu_1 p^2} a_2^{i^2 l r p}$ . Comparing indexes of  $b^{p^2}$  and  $a_2^p$  in two sides, we have  $i^2 = 1$  and  $r' = ir$ .

Since  $[\bar{a}_2^\theta, \bar{b}^\theta, \bar{b}^\theta] = (\bar{a}_2^{\nu_2 p})^\theta = (\bar{a}_2^\theta)^{\nu_2 p}$ , we have  $[a_2^{il}, b^l, b^l] = a_2^{il \nu_2 p}$ . Left side of above equation is  $a_2^{il^3 \nu_2 p}$ . Comparing index of  $a_2^p$  in two sides, we have  $l^2 = 1$ .

By calculation we have

$$\bar{a}_2^\theta = [\bar{a}_1, \bar{b}]^\theta = [a_1^i a_2^j b^{kp} y, b^l a_1^m] \equiv [a_1^i a_2^j, b^l] \pmod{Z(G) \cap G'}.$$

Moreover, by calculation we have

$$\bar{a}_2^\theta \equiv a_2^i a_1^{j \nu_2 p} \pmod{Z(G) \cap G'} \text{ for } l = 1$$

and

$$\bar{a}_2^\theta \equiv a_2^{-i} a_1^{(i-j) \nu_2 p} \pmod{Z(G) \cap G'} \text{ for } l = -1.$$

We have  $[\bar{a}_2^\theta, \bar{b}^\theta] = (\bar{a}_1^{\nu_2 p} \bar{a}_2^{s' p})^\theta = (\bar{a}_1^\theta)^{\nu_2 p} (\bar{a}_2^\theta)^{s' p}$ .

If  $l = 1$ , then we have  $[a_2^i a_1^{j \nu_2 p}, b a_1^m] = (a_1^i a_2^j b^{kp})^{\nu_2 p} a_2^{is' p}$ . Left side of above equation is  $b^{im \nu_1 p^2} a_2^{im r p} a_1^{i \nu_2 p} a_2^{is p} a_2^{j \nu_2 p}$ . Comparing indexes of  $b^{p^2}$ ,  $a_1^p$  and  $a_2^p$  in two sides, we have  $s' = s + ik \nu_2 \nu_1^{-1} r$ .

If  $l = -1$ , then we have  $[a_2^{-i} a_1^{(i-j) \nu_2 p}, b^{-1} a_1^m] = (a_1^i a_2^j b^{kp})^{\nu_2 p} a_2^{-is' p}$ . Left side of above equation is  $b^{-im \nu_1 p^2} a_2^{-im r p} a_1^{i \nu_2 p} a_2^{is p} a_2^{-i \nu_2 p} a_2^{(j-i) \nu_2 p}$ . Comparing indexes of  $b^{p^2}$ ,  $a_1^p$  and  $a_2^p$  in two sides, we have  $s' + s = 2 \nu_2 + ik \nu_2 \nu_1^{-1} r$ .

On the other hand, if there exists  $i, k$  such that  $i^2 = 1$ ,  $r' = ir$  and  $s' + s = 2 \nu_2 + ik \nu_2 \nu_1^{-1} r$ , then,  $\theta : \bar{a}_1 \rightarrow a_1^i b^{kp}$ ,  $\bar{b} \rightarrow b^{-1} a_1^{-ik \nu_2 \nu_1^{-1}}$  is an isomorphism from  $\bar{G}$  to  $G$ .

By above argument, if  $r = 0$ , then  $G$  is a group of Type (i), and if  $r \neq 0$ , then  $G$  is a group of Type (ii).

(3) By calculation we have  $[a_1^p, b] = [a_1, b]^p [a_1, b, a_1]^{\binom{p}{2}} [a_1, b, a_1, a_1]^{\binom{p}{3}} = a_2^p$ .



(4) Let  $N$  be a maximal subgroup of  $G$  such that  $N \neq M$ . Then we may assume  $N = \langle ba_1^i, \Phi(G) \rangle$ . By calculation we have  $[a_2, ba_1^i, ba_1^i] = [a_1^{\nu_1 p}, ba_1^i] = a_2^{\nu_1 p}$ . Hence  $N = \langle ba_1^i, a_2 \rangle$  is isomorphic to a group of Type (6) of Lemma 2.5. Thus  $N \in \mathcal{A}_2$ .

(5) By calculation,  $[a_1, b^3] = [a_1, b]^3[a_1, b, b]^3[a_1, b, b, b] = a_2^3[a_1^{-3}, b] = 1$ . Since  $M' = \langle [a_1, b^3], [a_2, a_1] \rangle$ ,  $|M'| = 3$ . Moreover,  $M = \langle a_2, a_1 \rangle * \langle b^{3\nu_1} a_2^r \rangle$ . By Lemma 3.5 (2) we get  $M \in \mathcal{A}_2$ . By (4) we get  $G \in \mathcal{A}_3$ .

(6) By calculation,  $[a_1, b^3] = [a_1, b]^3[a_1, b, b]^3[a_1, b, b, b] = a_2^3[a_1^3, b] = a_2^{-3}$ . Since  $M' = \langle [a_1, b^3], [a_2, a_1] \rangle$ ,  $|M'| = 9$ . By (4) we get  $G \in \mathcal{A}_3$  if and only if  $M \in \mathcal{A}_2$ . Moreover, we claim that  $M \in \mathcal{A}_2$  if and only if  $r^2 + 4\nu_1$  is not a square.

If  $M \in \mathcal{A}_2$ , then  $\langle a_2 a_1^x, b^3 a_1^y, \Phi(M) \rangle$  is either abelian or minimal non-abelian, where  $x, y \in F_3$ . For  $x \neq 0$  or  $y \neq 0$ , since  $[a_2 a_1^x, b^3 a_1^y] = b^{9y\nu_1} a_2^{3yr} a_2^{-3x} \neq 1$ , we have  $\langle a_2 a_1^x, b^3 a_1^y \rangle$  is minimal non-abelian. Hence  $\Phi(M) \leq \langle a_2 a_1^x, b^3 a_1^y \rangle$ . Moreover,  $\Phi(M) =$

$\langle a_1^{3x} a_2^3, a_1^{3y} b^9, a_2^{3(yr-x)} b^{9y\nu_1} \rangle$ . It follows that the equation  $\begin{vmatrix} x & 1 & 0 \\ y & 0 & 1 \\ 0 & yr-x & y\nu_1 \end{vmatrix} = 0$  only

has solution  $x = y = 0$ . That is, the equation  $x^2 - xyr - y^2\nu_1 = 0$  only has solution  $x = y = 0$ . Hence  $r^2 + 4\nu_1$  is not a square. Conversely, if  $r^2 + 4\nu_1$  is not a square, then, by above argument,  $\langle a_2 a_1^x, b^3 a_1^y, \Phi(M) \rangle$  is either abelian or minimal non-abelian. It is easy to see that  $\langle a_1, b^3, \Phi(M) \rangle$  and  $\langle a_1, a_2 b^{3z}, \Phi(M) \rangle$  are minimal non-abelian. Hence all maximal subgroups of  $M$  are abelian or minimal non-abelian. That is,  $M \in \mathcal{A}_2$ .

(7) By calculation,  $[a_1, b^p] = [a_1, b]^p = a_2^p$ . Since  $M' = \langle [a_1, b^p], [a_2, a_1] \rangle$ ,  $|M'| = p^2$ . By (4) we get  $G \in \mathcal{A}_3$  if and only if  $M \in \mathcal{A}_2$ . Moreover, we claim that  $M \in \mathcal{A}_2$  if and only if  $r^2 - 4\nu_1$  is not a square.

If  $M \in \mathcal{A}_2$ , then  $\langle a_2 a_1^x, b^p a_1^y, \Phi(M) \rangle$  is either abelian or minimal non-abelian, where  $x, y \in F_p$ . For  $x \neq 0$  or  $y \neq 0$ , since  $[a_2 a_1^x, b^p a_1^y] = b^{y\nu_1 p^2} a_2^{yrp} a_2^{xp} \neq 1$ , we have  $\langle a_2 a_1^x, b^p a_1^y \rangle$  is minimal non-abelian. Hence  $\Phi(M) \leq \langle a_2 a_1^x, b^p a_1^y \rangle$ . Moreover,  $\Phi(M) =$

$\langle a_1^{xp} a_2^p, a_1^{yp} b^{p^2}, a_2^{(yr+x)p} b^{y\nu_1 p} \rangle$ . It follows that the equation  $\begin{vmatrix} x & 1 & 0 \\ y & 0 & 1 \\ 0 & yr+x & y\nu_1 \end{vmatrix} = 0$  only

has solution  $x = y = 0$ . That is, the equation  $x^2 + xyr + y^2\nu_1 = 0$  only has solution  $x = y = 0$ . Hence  $r^2 - 4\nu_1$  is not a square. Conversely, if  $r^2 - 4\nu_1$  is not a square, then, by above argument,  $\langle a_2 a_1^x, b^p a_1^y, \Phi(M) \rangle$  is either abelian or minimal non-abelian. It is easy to see that  $\langle a_1, b^p, \Phi(M) \rangle$  and  $\langle a_1, a_2 b^{zp}, \Phi(M) \rangle$  are minimal non-abelian. Hence all maximal subgroups of  $M$  are abelian or minimal non-abelian. That is,  $M \in \mathcal{A}_2$ .  $\square$

**Lemma 5.11.** Suppose that  $G(\nu, t) = \langle b, a_1; a_2 \mid a_1^{p^2} = a_2^{p^2} = 1, b^{p^2} = a_2^{tp}, [a_1, b] = a_2, [a_2, a_1] = a_2^p, [a_2, b] = a_1^{\nu p} \rangle$ , where  $t \in F_p$ ,  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ .

- (1) If  $G = G(\nu, t)$ , then  $|G| = p^6$ ,  $G' = \langle a_1^p, a_2 \rangle$ ,  $G_3 = \langle a_1^p, a_2^p \rangle$ ,  $G_4 = \langle a_2^p \rangle$ ,  $G \in \mathcal{A}_3$ , and  $M = \langle b^p, a_1, a_2 \rangle$  is the unique three-generator maximal subgroup of  $G$ ;

(2)  $G(\nu, t) \cong G(\nu', t')$  if and only if  $(\nu', t') = (\nu, t)$ .

**Proof** (1) Let

$$K = \langle a_2, a_1 \mid a_2^{p^2} = a_1^{p^2} = 1, [a_2, a_1] = a_2^p \rangle.$$

We define an automorphism  $\beta$  of  $K$  as follows:

$$a_2^\beta = a_2 a_1^{\nu p}, \quad a_1^\beta = a_1 a_2.$$

Then  $a_2^{\beta p} = a_2$ ,  $a_1^{\beta p^2} = a_1$ ,  $o(\beta) \leq p^2$ . It follows from the cyclic extension theory that  $G = \langle K, b \rangle$  is a cyclic extension of  $K$  by  $C_{p^2}$ . Hence  $|G| = p^6$ . It is easy to check that  $G' = \langle a_1^p, a_2 \rangle$ ,  $G_3 = \langle a_1^p, a_2^p \rangle$ ,  $G_4 = \langle a_2^p \rangle$  and  $M = \langle b^p, a_1, a_2 \rangle$  is the unique three-generator maximal subgroup of  $G$ .

(2) For convenience, let  $G = \langle b, a_1, a_2 \rangle \cong G(\nu, t)$ ,  $\bar{G} = \langle \bar{b}, \bar{a}_1, \bar{a}_2 \rangle \cong G(\nu', t')$  and  $\theta$  be an isomorphism from  $\bar{G}$  to  $G$ . Since  $M = \langle b^p, a_1, a_2 \rangle$  and  $\bar{M} = \langle \bar{b}^p, \bar{a}_1, \bar{a}_2 \rangle$  are the unique three-generator maximal subgroups of  $G$  and  $\bar{G}$  respectively, we have  $\bar{M}^\theta = M$ . Hence we may assume that  $\bar{b}^\theta = b^l x$  and  $\bar{a}_1^\theta = a_1^i y$ , where  $l, i \in F_p^*$  and  $x \in M$ ,  $y \in \Phi(G)$ .

By calculation,  $\bar{a}_2^\theta = [\bar{a}_1, \bar{b}]^\theta \equiv a_2^{il} \pmod{G_3}$ .

Since  $[\bar{a}_2^\theta, \bar{a}_1^\theta] = (\bar{a}_2^p)^\theta = (\bar{a}_2^p)^\theta$ , we have  $[a_2^{il}, a_1^i] = a_2^{ilp}$ . Left side of above equation is  $a_2^{i^2 lp}$ . Comparing index of  $a_2^p$  in two sides, we have  $i = 1$ .

Since  $(\bar{b}^\theta)^{p^2} = (\bar{b}^{p^2})^\theta = (\bar{a}_2^{t'p})^\theta = (\bar{a}_2^\theta)^{t'p}$ , we have  $(b^l)^{p^2} = a_2^{t'lp}$ . Left side of above equation is  $a_2^{ltp}$ . Comparing index of  $a_2^p$  in two sides, we have  $t' = t$ .

Since  $[\bar{a}_2^\theta, \bar{b}^\theta, \bar{b}^\theta] = (\bar{a}_2^{\nu'p})^\theta = (\bar{a}_2^\theta)^{\nu'p}$ , we have  $[a_2^l, b^l, b^l] = a_2^{\nu'lp}$ . Left side of above equation is  $a_2^{l^3 \nu p}$ . Comparing index of  $a_2^p$  in two sides, we have  $\nu' = \nu$ .  $\square$

**Lemma 5.12.** Suppose that  $G(\nu, s, t) = \langle b, a_1, a_2 \mid a_1^{p^2} = a_2^{p^2} = 1, b^{p^m} = a_2^{tp}, [a_1, b] = a_2, [a_2, a_1] = 1, [a_2, b] = a_1^{\nu p} a_2^{sp} \rangle$ , where  $m \geq 2$ ,  $t \in F_p$ ,  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ .

(1) If  $G = G(\nu, s, t)$ , then  $|G| = p^{m+4}$ ,  $G' = \langle a_1^p, a_2 \rangle$ ,  $G_3 = \langle a_1^p, a_2^p \rangle$ ,  $G_4 = \langle a_2^p \rangle$ ,  $G \in \mathcal{A}_3$ ,  $M = \langle b^p, a_1, a_2 \rangle$  is the unique three-generator maximal subgroup of  $G$ ;

(2)  $G$  is one of the following non-isomorphic groups:

- (i)  $\langle b, a_1, a_2 \mid a_1^{p^2} = a_2^{p^2} = b^{p^m} = 1, [a_1, b] = a_2, [a_2, a_1] = 1, [a_2, b] = a_1^{\nu p} a_2^{sp} \rangle$ , where  $m \geq 2$ ,  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ ,  $s = \nu, \nu + 1, \dots, \nu + \frac{p-1}{2}$ ;
- (ii)  $\langle b, a_1, a_2 \mid a_1^{p^2} = a_2^{p^2} = 1, b^{p^m} = a_2^p, [a_1, b] = a_2, [a_2, a_1] = 1, [a_2, b] = a_1^{\nu p} \rangle$ , where  $m \geq 2$ ,  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ .

(3) If  $p = 3$  and  $\nu = -1$ , then  $M$  is abelian. If  $\nu = 1$  or  $p \geq 5$ , then  $|M'| = p$ .

**Proof** (1) Let

$$K = \langle a_2, a_1 \mid a_2^{p^2} = a_1^{p^2} = 1, [a_2, a_1] = a_2^p \rangle.$$

We define an automorphism  $\beta$  of  $K$  as follows:

$$a_2^\beta = a_2 a_1^{\nu p} a_2^{sp}, \quad a_1^\beta = a_1 a_2.$$

Then  $a_2^{\beta p} = a_2$ ,  $a_1^{\beta p^2} = a_1$ ,  $o(\beta) \leq p^2$ . It follows from the cyclic extension theory that  $G = \langle K, b \rangle$  is a cyclic extension of  $K$  by  $C_{p^m}$ . Hence  $|G| = p^{m+4}$ . It is easy to verify that  $G' = \langle a_1^p, a_2 \rangle$ ,  $G_3 = \langle a_1^p, a_2^p \rangle$ ,  $G_4 = \langle a_2^p \rangle$ ,  $G \in \mathcal{A}_3$  and  $M = \langle b^p, a_1, a_2 \rangle$  is the unique three-generator maximal subgroup of  $G$ .

(2) For convenience, let  $G = \langle b, a_1, a_2 \rangle \cong G(\nu, s, t)$ ,  $\bar{G} = \langle \bar{b}, \bar{a}_1, \bar{a}_2 \rangle \cong G(\nu', s', t')$  and  $\theta$  be an isomorphism from  $\bar{G}$  to  $G$ . Since  $M = \langle b^p, a_1, a_2 \rangle$  and  $\bar{M} = \langle \bar{b}^p, \bar{a}_1, \bar{a}_2 \rangle$  are the unique three-generator maximal subgroups of  $G$  and  $\bar{G}$  respectively, we have  $\bar{M}^\theta = M$ . Hence we may assume that  $\bar{b}^\theta = b^l x$  and  $\bar{a}_1^\theta = a_1^i a_2^j b^{kp^{m-1}} y$ , where  $l, i \in F_p^*$  and  $x \in M$ ,  $y \in G_3$ .

By calculation we have  $\bar{a}_2^\theta = [\bar{a}_1, \bar{b}]^\theta \equiv a_2^{il} \pmod{G_3}$ .

Since  $(\bar{b}^\theta)^{p^m} = (\bar{b}^{p^m})^\theta = (\bar{a}_2^{\theta})^{t'p} = (a_2^\theta)^{t'p}$ , we have  $(b^l x)^{p^m} = a_2^{t'ip}$ . Left side of the equation is  $a_2^{ltp}$ . By comparing index of  $a_2^3$  in two sides we have  $t'i = t$ .

Since  $[\bar{a}_2^\theta, \bar{b}^\theta, \bar{b}^\theta] = (\bar{a}_2^{\nu'p})^\theta = (\bar{a}_2^\theta)^{\nu'p}$ , we have  $[a_2^l, b^l, b^l] = a_2^{\nu'lp}$ . Left side of the equation is  $a_2^{l^3\nu p}$ . By comparing index of  $a_2^2$  in two sides we have  $\nu' = \nu$  and  $l^2 = 1$ .

By calculation we have  $\bar{a}_2^\theta = [\bar{a}_1, \bar{b}]^\theta = [a_1^i a_2^j b^{kp^{m-1}} y, b^l] \equiv [a_1^i a_2^j, b^l] \pmod{G_4}$ . Moreover, by calculation we have  $\bar{a}_2^\theta \equiv a_2^i a_1^{j\nu p} \pmod{G_4}$  for  $l = 1$  and  $\bar{a}_2^\theta \equiv a_2^{-i} a_1^{(i-j)\nu p} \pmod{G_4}$  for  $l = -1$ .

If  $l = 1$ , then

$$[a_2^i a_1^{j\nu p}, b] = [\bar{a}_2^\theta, \bar{b}^\theta] = (\bar{a}_1^{\nu p} \bar{a}_2^{s'p})^\theta = (\bar{a}_1^\theta)^{\nu p} (\bar{a}_2^\theta)^{s'p} = (a_1^i a_2^j b^{kp^{m-1}})^{\nu p} a_2^{is'p}.$$

Left side of the equation is  $a_1^{\nu p} a_2^{isp} a_2^{j\nu p}$ . By comparing indexes of  $a_1^p$  and  $a_2^p$  in two sides we have  $s' = s - i^{-1}tk\nu$ . On the other hand, if there exists  $i, k$  such that  $t'i = t$  and  $s' = s - i^{-1}tk\nu$ , then,  $\theta : \bar{a}_1 \rightarrow a^i b^{kp^{m-1}}$ ,  $\bar{b} \rightarrow b$  is an isomorphism from  $\bar{G}$  to  $G$ .

If  $l = -1$ , then we have

$$[a_2^{-i} a_1^{(i-j)\nu p}, b^{-1}] = [\bar{a}_2^\theta, \bar{b}^\theta] = (\bar{a}_1^{\nu p} \bar{a}_2^{s'p})^\theta = (\bar{a}_1^\theta)^{\nu p} (\bar{a}_2^\theta)^{s'p} = (a_1^i a_2^j b^{kp^{m-1}})^{\nu p} a_2^{-is'p}.$$

Left side of the equation is  $a_1^{\nu p} a_2^{isp} a_2^{-i\nu p} a_2^{(j-i)\nu p}$ . By comparing indexes of  $a_1^p$  and  $a_2^p$  in two sides we have  $s' + s = 2\nu + i^{-1}tk\nu$ . On the other hand, if there exists  $i, k$  such that  $t'i = t$  and  $s' + s = 2\nu + i^{-1}tk\nu$ , then,  $\theta : \bar{a}_1 \rightarrow a^i b^{kp^{m-1}}$ ,  $\bar{b} \rightarrow b^{-1}$  is an isomorphism from  $\bar{G}$  to  $G$ .

By above argument we get that  $G(\nu, s, 0) \cong G(\nu', s', t')$  if and only if  $\nu = \nu'$ ,  $t' = 0$  and  $s = s'$  or  $s + s' = 2\nu$ . It follows that  $G$  is a group of Type (i) for  $t = 0$ . If  $t \neq 0$ , then, by above argument,  $G(\nu, s, t) \cong G(\nu, 0, 1)$ . Hence  $G$  is a group of Type (ii).

(3) By calculation,  $[a_1^{\nu p}, b] = a_2^{\nu p}$ . Hence

$$[a_1, b^3] = [a_1, b]^3 [a_1, b, b]^3 [a_1, b, b, b] = a_2^3 [a_1^{3\nu}, b] = a_2^{3(1+\nu)}.$$

If  $p \geq 5$ , then  $[a_1, b^p] = a_2^p$ . Since  $M' = \langle [a_1, b^p] \rangle$ , the result is proved.  $\square$

**Theorem 5.13.** *Suppose that  $G$  is an  $\mathcal{A}_3$ -group all of whose maximal subgroups are  $\mathcal{A}_2$ -groups and there exists a three-generator maximal subgroup in  $G$ , and  $M' \leq Z(G)$  for every three-generator maximal subgroup  $M$ . Then  $d(G) = 2$  if and only if  $G$  is isomorphic to one of the following pairwise non-isomorphic groups:*

(Mi)  $G' \cong C_{p^2}$ . In this case,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1 + p)$  and  $\alpha_1(G) = p^3 + p^2$ .

(M1)  $\langle a, b, c \mid a^8 = 1, c^2 = a^4 = b^4, [a, b] = c, [c, a] = 1, [c, b] = 1 \rangle$ ; where  $|G| = 2^6$ ,  $c(G) = 2$ ,  $\Phi(G) = \langle a^2, b^2, c \rangle \cong C_4 \times C_2^2$ ,  $G' = \langle c \rangle$  and  $Z(G) = \langle c \rangle \cong C_4$ .

(M2)  $\langle a, b, c \mid a^8 = b^4 = 1, c^2 = a^4, [a, b] = c, [c, a] = 1, [c, b] = 1 \rangle$ ; moreover,  $|G| = 2^6$ ,  $c(G) = 2$ ,  $\Phi(G) = \langle a^2, b^2, c \rangle \cong C_4 \times C_2^2$ ,  $G' = Z(G) = \langle c \rangle \cong C_4$ .

(M3)  $\langle a, b, c \mid a^{p^3} = b^{p^2} = 1, c^p = a^{p^2}, [a, b] = c, [c, a] = 1, [c, b] = c^{tp} \rangle$ , where  $p > 2$  and  $t \in F_p^*$ ; moreover,  $|G| = p^6$ ,  $c(G) = 3$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^2} \times C_p^2$ ,  $G' = \langle c \rangle$  and  $Z(G) = \langle c^p \rangle \cong C_p$ .

(M4)  $\langle a, b, c \mid a^{p^3} = b^{p^2} = 1, c^p = a^{p^2}, [a, b] = c, [c, a] = c^p, [c, b] = 1 \rangle$ , where  $p > 2$ ; moreover,  $|G| = p^6$ ,  $c(G) = 3$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^2} \times C_p^2$ ,  $G' = \langle c \rangle$  and  $Z(G) = \langle a^{p^2} \rangle \cong C_p$ .

(M5)  $\langle a, b, c \mid a^{p^{n+1}} = b^{p^n} = 1, c^p = a^{p^n}, [a, b] = c, [c, a] = 1, [c, b] = 1 \rangle$ , where  $n \geq 3$  for  $p = 2$  and  $n \geq 2$ ; moreover,  $|G| = p^{2n+2}$ ,  $c(G) = 2$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^n} \times C_{p^{n-1}} \times C_p$ ,  $G' = \langle c \rangle$ ,  $Z(G) = \langle a^{p^2}, b^{p^2}, c \rangle \cong C_{p^{n-1}} \times C_{p^{n-2}} \times C_p$  if  $n > 2$ ,  $Z(G) = \langle a^{p^2}, b^{p^2}, c \rangle \cong C_p^2$  if  $n = 2$ .

(M6)  $\langle a, b, c \mid a^{p^2} = b^{p^2} = c^{p^2} = 1, [a, b] = c, [c, a] = c^p, [c, b] = 1 \rangle$ , where  $p > 2$ ; moreover,  $|G| = p^6$ ,  $c(G) = 3$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^2} \times C_p^2$ ,  $G' = \langle c \rangle$ ,  $Z(G) = \langle c^p \rangle \cong C_p$ .

(M7)  $\langle a, b, c \mid a^{p^n} = b^{p^n} = c^{p^2} = 1, [a, b] = c, [c, a] = 1, [c, b] = 1 \rangle$ , where  $n \geq 3$  for  $p = 2$  and  $n \geq 2$ ; moreover,  $|G| = p^{2n+2}$ ,  $c(G) = 2$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^{n-1}}^2 \times C_{p^2}$ ,  $G' = \langle c \rangle$ ,  $Z(G) = \langle a^{p^2}, b^{p^2}, c \rangle \cong C_{p^{n-2}}^2 \times C_{p^2}$  if  $n > 2$ ,  $Z(G) \cong C_{p^2}$  if  $n = 2$ .

(M8)  $\langle a, b, c \mid a^{p^{n+1}} = b^{p^2} = 1, c^p = a^{p^n}, [a, b] = c, [c, a] = c^p, [c, b] = 1 \rangle$ , where  $p > 2$  and  $n > 2$ ; moreover,  $|G| = p^{n+4}$ ,  $c(G) = 3$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^n} \times C_p^2$ ,  $G' = \langle c \rangle$  and  $Z(G) = \langle a^{p^2} \rangle \cong C_{p^{n-1}}$ .

(M9)  $\langle a, b, c \mid a^{p^n} = b^{p^3} = 1, c^p = b^{p^2}, [a, b] = c, [c, a] = c^{tp}, [c, b] = 1 \rangle$ , where  $p > 2$ ,  $n > 2$  and  $t \in F_p^*$ ; moreover,  $|G| = p^{n+4}$ ,  $c(G) = 3$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^{n-1}} \times C_{p^2} \times C_p$ ,  $G' = \langle c \rangle$  and  $Z(G) = \langle a^{p^2}, b^{p^2} \rangle \cong C_{p^{n-2}} \times C_p$ .

- (M10)  $\langle a, b; c \mid a^{p^n} = b^{p^2} = c^{p^2} = 1, [a, b] = c, [c, a] = c^p, [c, b] = 1 \rangle$ , where  $p > 2$  and  $n > 2$ . Moreover,  $|G| = p^{n+4}$ ,  $c(G) = 3$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^{n-1}} \times C_{p^2} \times C_p$ ,  $G' = \langle c \rangle$  and  $Z(G) = \langle a^{p^2}, c^p \rangle \cong C_{p^{n-2}} \times C_p$ .
- (M11)  $\langle a, b; c \mid a^{p^{n+1}} = b^{p^m} = 1, c^p = a^{p^n}, [a, b] = c, [c, a] = 1, [c, b] = 1 \rangle$ , where  $n > m \geq 2$ ; moreover,  $|G| = p^{n+m+2}$ ,  $c(G) = 2$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^n} \times C_{p^{m-1}} \times C_p$ ,  $G' = \langle c \rangle$ ,  $Z(G) = \langle a^{p^2}, b^{p^2}, c \rangle \cong C_{p^{n-1}} \times C_{p^{m-2}} \times C_p$  if  $m > 2$ ,  $Z(G) = \langle a^{p^2}, b^{p^2}, c \rangle \cong C_{p^{n-1}} \times C_p$  if  $m = 2$ .
- (M12)  $\langle a, b; c \mid a^{p^n} = b^{p^{m+1}} = 1, c^p = b^{p^m}, [a, b] = c, [c, a] = 1, [c, b] = 1 \rangle$ , where  $n > m \geq 2$ ;  $|G| = p^{n+m+2}$ ,  $c(G) = 2$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^{n-1}} \times C_{p^m} \times C_p$ ,  $G' = \langle c \rangle$  and  $Z(G) = \langle a^{p^2}, b^{p^2}, c \rangle \cong C_{p^{n-2}} \times C_{p^{m-1}} \times C_p$ .
- (M13)  $\langle a, b; c \mid a^{p^n} = b^{p^m} = c^{p^2} = 1, [a, b] = c, [c, a] = 1, [c, b] = 1 \rangle$ , where  $n > m \geq 2$ ; moreover,  $|G| = p^{n+m+2}$ ,  $c(G) = 2$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^{n-1}} \times C_{p^{m-1}} \times C_{p^2}$ ,  $G' = \langle c \rangle$ ,  $Z(G) = \langle a^{p^2}, b^{p^2}, c \rangle \cong C_{p^{n-2}} \times C_{p^{m-2}} \times C_{p^2}$  if  $m > 2$ ,  $Z(G) \cong C_{p^{n-2}} \times C_{p^2}$  if  $m = 2$ .
- (Mii)  $c(G) = 3$  and  $G' \cong C_p^2$ . In this case,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1 + p)$  and  $\alpha_1(G) = p^3 + p^2$ .
- (M14)  $\langle a, b; c \mid a^8 = b^4 = c^2 = 1, [a, b] = c, [c, a] = 1, [c, b] = a^4 \rangle$ ; where  $|G| = 2^6$ ,  $\Phi(G) = \langle a^2, b^2, c \rangle \cong C_4 \times C_2^2$ ,  $G' = \langle a^4, c \rangle$  and  $Z(G) = \langle a^2 \rangle \cong C_4$ .
- (M15)  $\langle a, b; c, d \mid a^4 = b^4 = c^2 = d^2 = 1, [a, b] = c, [c, a] = d, [c, b] = 1, [d, a] = [d, b] = 1 \rangle$ ; moreover,  $|G| = 2^6$ ,  $\Phi(G) = \langle a^2, b^2, c, d \rangle \cong C_2^4$ ,  $G' = \langle c, d \rangle$  and  $Z(G) = \langle b^2, d \rangle \cong C_2^2$ .
- (M16)  $\langle a, b; c \mid a^8 = b^4 = c^2 = 1, [a, b] = c, [c, a] = a^4, [c, b] = 1 \rangle$ , where  $|G| = 2^6$ ,  $\Phi(G) = \langle a^2, b^2, c \rangle \cong C_4 \times C_2^2$ ,  $G' = \langle a^4, c \rangle$  and  $Z(G) = \langle a^4, b^2 \rangle \cong C_2^2$ .
- (M17)  $\langle a, b; c \mid a^{2^{n+1}} = b^4 = c^2 = 1, [a, b] = c, [c, a] = a^{2^n}, [c, b] = 1 \rangle$ , where  $n > 2$ ; moreover,  $|G| = 2^{n+4}$ ,  $\Phi(G) = \langle a^2, b^2, c \rangle \cong C_{2^n} \times C_2^2$ ,  $G' = \langle a^{2^n}, c \rangle$  and  $Z(G) = \langle a^4, b^2 \rangle \cong C_{2^{n-1}} \times C_2$ .
- (M18)  $\langle a, b; c \mid a^{2^n} = b^8 = c^2 = 1, [a, b] = c, [c, a] = b^4, [c, b] = 1 \rangle$ , where  $n > 2$ ; moreover,  $|G| = 2^{n+4}$ ,  $\Phi(G) = \langle a^2, b^2, c \rangle \cong C_{2^{n-1}} \times C_4 \times C_2$ ,  $G' = \langle b^4, c \rangle$  and  $Z(G) = \langle a^4, b^2 \rangle \cong C_{2^{n-2}} \times C_4$ .
- (M19)  $\langle a, b; c, d \mid a^{2^n} = b^4 = c^2 = d^2 = 1, [a, b] = c, [c, a] = d, [c, b] = 1, [d, a] = [d, b] = 1 \rangle$ , where  $n > 2$ ; moreover,  $|G| = 2^{n+4}$ ,  $\Phi(G) = \langle a^2, b^2, c, d \rangle \cong C_{2^{n-1}} \times C_2^3$ ,  $G' = \langle c, d \rangle$  and  $Z(G) = \langle a^4, b^2, d \rangle \cong C_{2^{n-2}} \times C_2^2$ .
- (Miii)  $G_3 \cong C_p$  and  $\Phi(G')G_3 \cong C_p^2$ . In this case,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1 + p)$  and  $\alpha_1(G) = p^3 + 2p^2$ .
- (M20)  $\langle a, b; c \mid a^8 = b^8 = 1, c^2 = b^4, [a, b] = c, [c, a] = 1, [c, b] = a^4b^4 \rangle$ ; where  $|G| = 2^7$ ,  $\Phi(G) = \langle a^2, b^2, c \rangle \cong C_4^2 \times C_2$ ,  $G' = \langle a^4, c \rangle$  and  $Z(G) = \langle a^4, c^2 \rangle \cong C_2^2$ .

- (M21)  $\langle a, b; c \mid a^8 = c^4 = 1, b^4 = c^2, [a, b] = c, [c, a] = a^4, [c, b] = 1 \rangle$ ; where  $|G| = 2^7$ ,  $\Phi(G) = \langle a^2, b^2, c \rangle \cong C_4^2 \times C_2$ ,  $G' = \langle a^4, c \rangle$  and  $Z(G) = \langle a^4, c^2 \rangle \cong C_2^2$ .
- (M22)  $\langle a, b; c \mid a^8 = c^4 = 1, b^4 = c^2, [a, b] = c, [c, a] = 1, [c, b] = a^4 \rangle$ ; where  $|G| = 2^7$ ,  $\Phi(G) = \langle a^2, b^2, c \rangle \cong C_4^2 \times C_2$ ,  $G' = \langle a^4, c \rangle$ ,  $Z(G) = \langle a^4, c^2 \rangle \cong C_2^2$ .
- (M23)  $\langle a, b; c \mid a^4 = b^8 = c^4 = 1, [a, b] = c, [c, a] = 1, [c, b] = b^4 \rangle$ ; where  $|G| = 2^7$ ,  $\Phi(G) = \langle a^2, b^2, c \rangle \cong C_4^2 \times C_2$ ,  $G' = \langle b^4, c \rangle$  and  $Z(G) = \langle b^4, c^2 \rangle \cong C_2^2$ .
- (M24)  $\langle a, b; c \mid a^{p^3} = b^{p^3} = 1, c^p = a^{p^2} b^{sp^2}, [a, b] = c, [c, a] = 1, [b, c] = b^{p^2} \rangle$ , where  $p > 2$ ,  $s \in F_p$  and  $1 + 4s \notin F_p^2$ ; moreover,  $|G| = p^7$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^2} \times C_{p^2} \times C_p$ ,  $G' = \langle b^{p^2}, c \rangle$  and  $Z(G) = \langle a^{p^2}, b^{p^2} \rangle \cong C_p^2$ .
- (M25)  $\langle a, b; c \mid a^{p^3} = b^{p^2} = c^{p^2} = 1, [a, b] = c, [c, a] = 1, [c, b] = c^{tp} a^{-tp^2} \rangle$ , where  $p > 2$  and  $t^2 - 4t \notin F_p^2$ ; moreover,  $|G| = p^7$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^2} \times C_{p^2} \times C_p$ ,  $G' = \langle a^{p^2}, c \rangle$  and  $Z(G) = \langle a^{p^2}, c^p \rangle \cong C_p^2$ .
- (M26)  $\langle a, b; c \mid a^{p^3} = b^{p^3} = 1, [a, b] = c, c^p = b^{p^2}, [c, a] = 1, [b, c] = a^{\nu p^2} \rangle$ , where  $p > 2$ ,  $\nu = 1$  or a fixed quadratic non-residue modular  $p$ , and  $-\nu \notin F_p^2$ ; moreover,  $|G| = p^7$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^2} \times C_{p^2} \times C_p$ ,  $G' = \langle a^{p^2}, c \rangle$  and  $Z(G) = \langle a^{p^2}, b^{p^2} \rangle \cong C_p^2$ .
- (M27)  $\langle a, b; c \mid a^{p^3} = b^{p^2} = c^{p^2} = 1, [a, b] = c, [c, a] = 1, [b, c] = a^{\nu p^2} \rangle$ , where  $p > 2$ ,  $\nu = 1$  or a fixed quadratic non-residue modular  $p$ , and  $-\nu \notin F_p^2$ ; moreover,  $|G| = p^7$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^2} \times C_{p^2} \times C_p$ ,  $G' = \langle a^{p^2}, c \rangle$  and  $Z(G) = \langle a^{p^2}, c^p \rangle \cong C_p^2$ .
- (M28)  $\langle a, b; c \mid a^{p^2} = b^{p^3} = c^{p^2} = 1, [a, b] = c, [c, a] = 1, [c, b] = c^p b^{-p^2}, [b^{p^2}, a] = 1 \rangle$ , where  $p > 2$ ; moreover,  $|G| = p^7$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^2} \times C_{p^2} \times C_p$ ,  $G' = \langle b^{p^2}, c \rangle$  and  $Z(G) = \langle c^p, b^{p^2} \rangle \cong C_p^2$ .
- (M29)  $\langle a, b; c \mid a^{p^2} = b^{p^3} = c^{p^2} = 1, [a, b] = c, [c, a] = 1, [b, c] = b^{p^2} \rangle$ , where  $p > 2$ ; moreover,  $|G| = p^7$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^2} \times C_{p^2} \times C_p$ ,  $G' = \langle b^{p^2}, c \rangle$  and  $Z(G) = \langle c^p, b^{p^2} \rangle \cong C_p^2$ .
- (M30)  $\langle a, b; c \mid a^{p^{n+1}} = b^{p^3} = 1, [a, b] = c, c^p = a^{p^n}, [a, c] = b^{\nu p^2}, [b, c] = 1 \rangle$ , where  $n > 2$ ,  $\nu = 1$  or a fixed quadratic non-residue modular  $p$ ; moreover,  $|G| = p^{n+5}$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^n} \times C_{p^2} \times C_p$ ,  $G' = \langle b^{p^2}, c \rangle$  and  $Z(G) = \langle a^{p^2}, b^{p^2} \rangle \cong C_{p^{n-1}} \times C_p$ .
- (M31)  $\langle a, b; c \mid a^{p^{n+1}} = b^{p^3} = 1, [a, b] = c, c^p = a^{p^n} b^{s\eta p^2}, [a, c] = b^{\eta p^2}, [b, c] = 1 \rangle$ , where  $p > 2$ ,  $n > 2$ ,  $s = 1, 2, \dots, \frac{p-1}{2}$ , and  $\eta$  is a fixed quadratic non-residue modular  $p$ ; moreover,  $|G| = p^{n+5}$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^n} \times C_{p^2} \times C_p$ ,  $G' = \langle b^{p^2}, c \rangle$  and  $Z(G) = \langle a^{p^2}, b^{p^2} \rangle \cong C_{p^{n-1}} \times C_p$ .
- (M32)  $\langle a, b; c \mid a^{p^{n+1}} = b^{p^2} = c^{p^2} = 1, [a, b] = c, [c, b] = 1, [c, a] = c^p a^{-p^n}, [a^{p^n}, b] = 1 \rangle$ , where  $p > 2$  and  $n > 2$ ; moreover,  $|G| = p^{n+5}$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^n} \times C_{p^2} \times C_p$ ,  $G' = \langle a^{p^n}, c \rangle$  and  $Z(G) = \langle a^{p^2}, c^p \rangle \cong C_{p^{n-1}} \times C_p$ .

- (M33)  $\langle a, b; c \mid a^{p^{n+1}} = b^{p^3} = 1, [a, b] = c, c^p = b^{p^2}, [b, c] = 1, [a, c] = a^{p^n} \rangle$ , where  $n > 2$ ; moreover,  $|G| = p^{n+5}$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^n} \times C_{p^2} \times C_p$ ,  $G' = \langle a^{p^n}, c \rangle$  and  $Z(G) = \langle a^{p^2}, c^p \rangle \cong C_{p^{n-1}} \times C_p$ .
- (M34)  $\langle a, b; c \mid a^{p^{n+1}} = b^{p^2} = c^{p^2} = 1, [a, b] = c, [b, c] = 1, [a, c] = a^{p^n} \rangle$ , where  $n > 2$ ; moreover,  $|G| = p^{n+5}$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^n} \times C_{p^2} \times C_p$ ,  $G' = \langle a^{p^n}, c \rangle$  and  $Z(G) = \langle a^{p^2}, c^p \rangle \cong C_{p^{n-1}} \times C_p$ .
- (M35)  $\langle a, b; c \mid a^{p^n} = b^{p^3} = c^{p^2} = 1, [a, b] = c, [c, b] = 1, [c, a] = c^{tp}b^{-tp^2}, [b^{p^2}, a] = 1 \rangle$ , where  $p > 2$ ,  $n > 2$  and  $t^2 + 4t \notin F_p^2$ ; moreover,  $|G| = p^{n+5}$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^{n-1}} \times C_{p^2} \times C_{p^2}$ ,  $G' = \langle b^{p^2}, c \rangle$  and  $Z(G) = \langle a^{p^2}, b^{p^2}, c^p \rangle \cong C_{p^{n-2}} \times C_p \times C_p$ .
- (M36)  $\langle a, b; c \mid a^{p^n} = b^{p^3} = c^{p^2} = 1, [a, b] = c, [c, b] = 1, [a, c] = b^{\eta p^2} \rangle$ , where  $p > 2$ ,  $n > 2$ ,  $\eta$  is a fixed quadratic non-residue modular  $p$ ; moreover,  $|G| = p^{n+5}$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^{n-1}} \times C_{p^2} \times C_{p^2}$ ,  $G' = \langle b^{p^2}, c \rangle$  and  $Z(G) = \langle a^{p^2}, b^{p^2}, c^p \rangle \cong C_{p^{n-2}} \times C_p \times C_p$ .
- (Miv)  $\Phi(G') \leq G_3 \cong C_p^2$ . In this case,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1 + p)$  and  $\alpha_1(G) = p^3 + p^2$  except for (M37)–(M39).
- (M37)  $\langle a, b; c \mid a^8 = b^8 = c^2 = 1, [a, b] = c, [c, a] = a^4b^4, [c, b] = a^4, [a^4, b] = 1 \rangle$ ; where  $|G| = 2^7$ ,  $\Phi(G) = \langle a^2, b^2, c \rangle \cong C_4^2 \times C_2$ ,  $G' = \langle a^4, b^4, c \rangle \cong C_2^3$ ,  $Z(G) = \langle a^4, b^4 \rangle \cong C_2^2$ , and  $\alpha_1(G) = 18$ ;
- (M38)  $\langle a, b; c \mid a^8 = b^8 = 1, [a, b] = c, [c, a] = c^2 = b^4, [c, b] = a^4 \rangle$ ; where  $|G| = 2^7$ ,  $\Phi(G) = \langle a^2, b^2, c \rangle \cong C_4^2 \times C_2$ ,  $G' = \langle a^4, c \rangle \cong C_4 \times C_2$ ,  $Z(G) = \langle a^4, b^4 \rangle \cong C_2^2$ , and  $\alpha_1(G) = 14$ ;
- (M39)  $\langle a, b; c \mid a^8 = c^4 = 1, [a, b] = c, [c, a] = c^2, [c, b] = a^4 = b^4 \rangle$ ; where  $|G| = 2^7$ ,  $\Phi(G) = \langle a^2, b^2, c \rangle \cong C_4 \times C_4 \times C_2$ ,  $G' = \langle a^4, c \rangle \cong C_4 \times C_2$ ,  $Z(G) = \langle a^4, c^2 \rangle \cong C_2^2$ , and  $\alpha_1(G) = 18$ ;
- (M40)  $\langle a, b; c \mid a^{p^2} = b^{p^2} = c^p = 1, [a, b] = c, [c, a] = a^p, [c, b] = b^p \rangle$ , where  $p > 3$ ; moreover,  $|G| = p^5$ ,  $\Phi(G) = G' = \langle a^p, b^p, c \rangle \cong C_p^3$  and  $Z(G) = \langle a^p, b^p \rangle \cong C_p^2$ ;
- (M41)  $\langle a, b; c \mid a^{p^3} = b^{p^3} = c^p = 1, [a, b] = c, [c, a] = b^{\nu p^2}, [c, b] = a^{-p^2}, [a^{p^2}, b] = 1 \rangle$ , where  $p > 2$ ,  $\nu = 1$  or a fixed quadratic non-residue modular  $p$  such that  $-\nu \notin F_p^2$ ; moreover,  $|G| = p^7$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^2} \times C_{p^2} \times C_p$ ,  $G' = \langle a^{p^2}, b^{p^2}, c \rangle$  and  $Z(G) = \langle a^p, b^p \rangle \cong C_{p^2} \times C_{p^2}$ ;
- (M42)  $\langle a, b; c \mid a^{p^3} = b^{p^3} = c^p = 1, [a, b] = c, [c, a]^{1+r} = a^{p^2}b^{p^2}, [c, b]^{1+r} = a^{-rp^2}b^{p^2}, [a^{p^2}, b] = 1 \rangle$ , where  $p > 3$  and  $-r \notin (F_p)^2$ ; moreover,  $|G| = p^7$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^2} \times C_{p^2} \times C_p$ ,  $G' = \langle a^{p^2}, b^{p^2}, c \rangle$  and  $Z(G) = \langle a^p, b^p \rangle \cong C_{p^2} \times C_{p^2}$ ;
- (M43)  $\langle a, b; c, d, e \mid a^p = b^p = c^p = d^p = e^p = 1, [a, b] = c, [c, a] = d, [c, b] = e, [d, a] = [d, b] = [e, a] = [e, b] = 1 \rangle$ , where  $p > 3$ ; moreover,  $|G| = p^5$ ,  $\Phi(G) = G' = \langle c, d, e \rangle \cong C_p^3$  and  $Z(G) = \langle d, e \rangle \cong C_p^2$ ;

- (M44)  $\langle a, b, c \mid a^{p^3} = b^{p^3} = 1, [a, b] = c, [c, a] = c^p = b^{sp^2}, [c, b] = a^{-\nu p^2} b^{st\nu p^2} \rangle$ , where  $p > 2$ ,  $\nu = 1$  or a fixed quadratic non-residue modular  $p$ ,  $s \in F_p^*$ ,  $t = 0, 1, \dots, \frac{p-1}{2}$  such that  $(st\nu)^2 - 4\nu(s+1) \notin F_p^2$ ; moreover,  $|G| = p^7$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^2} \times C_{p^2} \times C_p$ ,  $G' = \langle a^{p^2}, c \rangle \cong C_{p^2} \times C_p$  and  $Z(G) = \langle a^{p^2}, b^{p^2} \rangle \cong C_p^2$ ;
- (M45)  $\langle a, b, c \mid a^{p^3} = b^{p^3} = 1, [a, b] = c, [c, a] = c^p = b^{-p^2}, [b, c] = a^{\nu p^2} \rangle$ , where  $p > 2$ ,  $\nu = 1$  or a fixed quadratic non-residue modular  $p$ ; moreover,  $|G| = p^7$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^2} \times C_{p^2} \times C_p$ ,  $G' = \langle a^{p^2}, c \rangle \cong C_{p^2} \times C_p$  and  $Z(G) = \langle a^{p^2}, b^{p^2} \rangle \cong C_p^2$ ;
- (M46)  $\langle a, b, c \mid a^{p^3} = b^{p^2} = c^{p^2} = 1, [a, b] = c, [c, a] = c^p, [c, b] = a^{-\nu p^2} c^{t\nu p} \rangle$ , where  $p > 2$ ,  $\nu = 1$  or a fixed quadratic non-residue modular  $p$ ,  $t = 0, 1, \dots, \frac{p-1}{2}$  such that  $(t\nu)^2 - 4\nu \notin F_p^2$ ; moreover,  $|G| = p^7$ ,  $\Phi(G) = \langle a^p, b^p, c \rangle \cong C_{p^2} \times C_{p^2} \times C_p$ ,  $G' = \langle a^{p^2}, c \rangle \cong C_{p^2} \times C_p$  and  $Z(G) = \langle a^{p^2}, c^p \rangle \cong C_p^2$ ;
- (M47)  $\langle a, b, c, d \mid a^{p^2} = b^{p^2} = c^p = d^p = 1, [a, b] = c, [c, a] = b^{\nu p}, [c, b] = d, [d, a] = [d, b] = 1 \rangle$ , where  $p > 2$ ,  $\nu = 1$  or a fixed quadratic non-residue modular  $p$ ; moreover,  $|G| = p^6$ ,  $\Phi(G) = \langle a^p, b^p, c, d \rangle \cong C_p^4$ ,  $G' = \langle b^p, c, d \rangle \cong C_p^3$  and  $Z(G) = \langle a^p, b^p, d \rangle \cong C_p^3$ .
- (Mv)  $c(G) = 4$ ,  $G$  has a unique three-generator maximal subgroup  $M$ , and  $G/M'$  is a group of Type (4) in Lemma 2.5. In this case,  $p \geq 5$ ,  $|M'| = p$ ,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1+p)$  and  $\alpha_1(G) = 2p^2$ .
- (M48)  $\langle b, a_1; a_2, a_3 \mid b^{p^2} = a_1^p = a_2^p = a_3^p = 1, [a_1, b] = a_2, [a_2, b] = a_3, [a_2, a_1] = b^p, [a_3, b] = b^p, [a_3, a_1] = 1 \rangle$ ; where  $|G| = p^5$ ,  $\Phi(G) = G' = \langle a_2, a_3, b^p \rangle \cong C_p^3$  and  $Z(G) = \langle b^p \rangle \cong C_p$ .
- (M49)  $\langle b, a_1; a_2, a_3 \mid b^{p^2} = a_1^p = a_2^p = a_3^p = 1, [a_1, b] = a_2, [a_2, b] = a_3, [a_2, a_1] = b^{\eta p}, [a_3, b] = b^p, [a_3, a_1] = 1 \rangle$ , where  $\eta$  is a fixed quadratic non-residue modulo  $p$ ; moreover,  $|G| = p^5$ ,  $\Phi(G) = G' = \langle a_2, a_3, b^p \rangle \cong C_p^3$  and  $Z(G) = \langle b^p \rangle \cong C_p$ .
- (M50)  $\langle b, a_1; a_2, a_3 \mid b^{p^2} = a_1^p = a_2^p = a_3^p = 1, [a_1, b] = a_2, [a_2, b] = a_3, [a_2, a_1] = b^p, [a_3, b] = b^{\eta p}, [a_3, a_1] = 1 \rangle$ , where  $p \equiv 1 \pmod{4}$  and  $\eta$  is a fixed quadratic non-residue modulo  $p$ ; moreover,  $|G| = p^5$ ,  $\Phi(G) = G' = \langle a_2, a_3, b^p \rangle \cong C_p^3$  and  $Z(G) = \langle b^p \rangle \cong C_p$ .
- (M51)  $\langle b, a_1; a_2, a_3 \mid b^{p^2} = a_1^p = a_2^p = a_3^p = 1, [a_1, b] = a_2, [a_2, b] = a_3, [a_2, a_1] = b^{\eta p}, [a_3, b] = b^{\eta p}, [a_3, a_1] = 1 \rangle$ , where  $p \equiv 1 \pmod{4}$  and  $\eta$  is a fixed quadratic non-residue modulo  $p$ ; moreover,  $|G| = p^5$ ,  $\Phi(G) = G' = \langle a_2, a_3, b^p \rangle \cong C_p^3$  and  $Z(G) = \langle b^p \rangle \cong C_p$ .
- (M52)  $\langle b, a_1; a_2, a_3 \mid b^p = a_1^{p^2} = a_2^p = a_3^p = 1, [a_1, b] = a_2, [a_2, b] = a_3, [a_2, a_1] = a_1^p, [a_3, b] = a_1^{\nu p}, [a_3, a_1] = 1 \rangle$ , where  $\nu = 1$ ,  $\eta_1$  or  $\eta_2$ ,  $\{1, \eta_1, \eta_2\}$  is a transver-



sal for  $(F_p^*)^3$  in  $F_p^*$ ; moreover,  $|G| = p^5$ ,  $\Phi(G) = G' = \langle a_2, a_3, a_1^p \rangle \cong C_p^3$  and  $Z(G) = \langle a_1^p \rangle \cong C_p$ .

$$(M53) \quad \langle b, a_1; a_2, a_3, a_4 \mid b^p = a_1^p = a_2^p = a_3^p = a_4^p = 1, [a_1, b] = a_2, [a_2, b] = a_3, [a_2, a_1] = a_4, [a_3, b] = a_4, [a_3, a_1] = [a_4, a_1] = [a_4, b] = 1 \rangle, \text{ where } |G| = p^5, \\ \Phi(G) = G' = \langle a_2, a_3, a_4 \rangle \cong C_p^3 \text{ and } Z(G) = \langle a_4 \rangle \cong C_p.$$

(Mvi)  $c(G) = 4$ ,  $G$  has a unique three-generator maximal subgroup  $M$ ,  $|M'| = 9$  and  $G/M'$  is a group of Type (6) in Lemma 2.5. In this case,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1+p)$  and  $\alpha_1(G) = 2p^2 + p$ .

$$(M54) \quad \langle b, a_1; a_2 \mid b^{27} = a_1^9 = a_2^9 = 1, [a_1, b] = a_2, [a_2, a_1] = b^{-9}, [a_2, b] = a_1^3 a_2^{3s} \rangle, \\ \text{where } s = 0, 2; \text{ moreover, } |G| = 3^7, \Phi(G) = \langle a_2, a_1^3, b^3 \rangle \cong C_3 \times C_9 \times C_9, \\ G' = \langle a_2, a_1^3, b^9 \rangle \cong C_3^2 \times C_9 \text{ and } Z(G) = \langle a_2^3, b^9 \rangle \cong C_3^2.$$

$$(M55) \quad \langle b, a_1; a_2 \mid b^{27} = a_1^9 = a_2^9 = 1, [a_1, b] = a_2, [a_2, a_1] = b^9 a_2^3, [a_2, b] = a_1^3 \rangle, \text{ where } \\ |G| = 3^7, \Phi(G) = \langle a_2, a_1^3, b^3 \rangle \cong C_3 \times C_9 \times C_9, G' = \langle a_2, a_1^3, b^9 \rangle \cong C_3^2 \times C_9 \\ \text{and } Z(G) = \langle a_2^3, b^9 \rangle \cong C_3^2.$$

(Mvii)  $c(G) = 4$ ,  $G$  has a unique three-generator maximal subgroup  $M$ ,  $|M'| = p^2$  where  $p \geq 5$  and  $G/M'$  is a group of Type (6) in Lemma 2.5. In this case,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1+p)$  and  $\alpha_1(G) = 2p^2 + p$ .

$$(M56) \quad \langle b, a_1; a_2 \mid b^{p^3} = a_1^{p^2} = a_2^{p^2} = 1, [a_1, b] = a_2, [a_2, a_1] = b^{\nu_1 p^2}, [a_2, b] = a_1^{\nu_2 p} a_2^{s p} \rangle, \\ \text{where } \nu_1, \nu_2 = 1 \text{ or a fixed quadratic non-residue modulo } p \text{ such that } -\nu_1 \text{ is} \\ \text{not a square, and } s = 2^{-1}\nu_2, 2^{-1}\nu_2 + 1, \dots, 2^{-1}\nu_2 + \frac{p-1}{2}; \text{ moreover, } |G| = p^7, \\ \Phi(G) = \langle a_2, a_1^p, b^p \rangle \cong C_{p^2} \times C_{p^2} \times C_p, G' = \langle a_2, a_1^p, b^{p^2} \rangle \cong C_p^2 \times C_{p^2} \text{ and} \\ Z(G) = \langle a_2^p, b^{p^2} \rangle \cong C_p^2.$$

$$(M57) \quad \langle b, a_1; a_2 \mid b^{p^3} = a_1^{p^2} = a_2^{p^2} = 1, [a_1, b] = a_2, [a_2, a_1] = b^{\nu_1 p^2} a_2^{rp}, [a_2, b] = \\ a_1^{\nu_2 p} \rangle, \text{ where } \nu_1, \nu_2 = 1 \text{ or a fixed quadratic non-residue modulo } p \text{ and } r = \\ 1, 2, \dots, \frac{p-1}{2} \text{ such that } r^2 - 4\nu_1 \text{ is not a square. Moreover, } |G| = p^7, \Phi(G) = \\ \langle a_2, a_1^p, b^p \rangle \cong C_{p^2} \times C_{p^2} \times C_p, G' = \langle a_2, a_1^p, b^{p^2} \rangle \cong C_p^2 \times C_{p^2} \text{ and } Z(G) = \\ \langle a_2^p, b^{p^2} \rangle \cong C_p^2.$$

(Mviii)  $c(G) = 4$ ,  $G$  has a unique three-generator maximal subgroup  $M$ ,  $|M'| = p$  where  $p \geq 3$  and  $G/M'$  is a group of Type (6) in Lemma 2.5. In this case,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1+p)$  and  $\alpha_1(G) = 2p^2$ .

$$(M58) \quad \langle b, a_1; a_2 \mid a_1^{p^2} = a_2^{p^2} = 1, b^{p^2} = a_2^{tp}, [a_1, b] = a_2, [a_2, a_1] = a_2^p, [a_2, b] = a_1^{\nu p} \rangle, \\ \text{where } t \in F_p, \nu = 1 \text{ or a fixed quadratic non-residue modulo } p; \text{ moreover, } \\ |G| = p^6, \Phi(G) = \langle a_2, a_1^p, b^p \rangle \cong C_{p^2} \times C_p^2, G' = \langle a_2, a_1^p \rangle \cong C_{p^2} \times C_p, Z(G) = \\ \langle a_2^p, b^{p^2} \rangle \cong C_p \text{ except for } p = 3 \text{ and } \nu = -1, \text{ In case of } p = 3 \text{ and } \nu = -1, \\ \text{we have } Z(G) = \langle a_2^p, b^p \rangle \cong C_p^2 \text{ if } t \equiv 0 \pmod{p} \text{ or } Z(G) = \langle a_2^p, b^p \rangle \cong C_{p^2} \text{ if} \\ t \not\equiv 0 \pmod{p}.$$

(M59)  $\langle b, a_1; a_2 \mid a_1^{p^2} = a_2^{p^2} = b^{p^m} = 1, [a_1, b] = a_2, [a_2, a_1] = 1, [a_2, b] = a_1^{\nu p} a_2^{s p} \rangle$ ,  
where  $m \geq 2$ ,  $\nu = 1$  for  $p = 3$ ,  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$  for  $p \geq 5$ ,  $s = \nu, \nu + 1, \dots, \nu + \frac{p-1}{2}$ ; moreover,  $|G| = p^{m+4}$ ,  
 $\Phi(G) = \langle a_2, a_1^p, b^p \rangle \cong C_p \times C_{p^{m-1}} \times C_{p^2}$ ,  $G' = \langle a_2, a_1^p \rangle \cong C_p \times C_{p^2}$ ,  $Z(G) = \langle a_2^p, b^{p^2} \rangle \cong C_p \times C_{p^{m-2}}$  for  $m > 2$ ,  $Z(G) = \langle a_2^p, b^{p^2} \rangle \cong C_p$  for  $m = 2$ .

(M60)  $\langle b, a_1; a_2 \mid a_1^{p^2} = a_2^{p^2} = 1, b^{p^m} = a_2^p, [a_1, b] = a_2, [a_2, a_1] = 1, [a_2, b] = a_1^{\nu p} \rangle$ ,  
where  $m \geq 2$ ,  $\nu = 1$  for  $p = 3$ ,  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$  for  $p \geq 5$ . Moreover,  $|G| = p^{m+4}$ ,  $\Phi(G) = \langle a_2, a_1^p, b^p \rangle \cong C_{p^m} \times C_p^2$ ,  
 $G' = \langle a_2, a_1^p \rangle \cong C_p \times C_{p^2}$ ,  $Z(G) = \langle a_2^p, b^{p^2} \rangle \cong C_{p^{m-2}}$  for  $m > 2$ ,  $Z(G) = \langle a_2^p, b^{p^2} \rangle \cong C_p$  for  $m = 2$ .

(Mix)  $c(G) = 4$ ,  $G$  has a unique three-generator maximal subgroup  $M$ ,  $|M'| = 3$  and  $G/M' \in \mathcal{A}_3$ . In this case,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1 + p)$  and  $\alpha_1(G) = 2p^2$ .

(M61)  $\langle b, a_1; a_2 \mid a_1^9 = a_2^9 = b^{27} = 1, [a_1, b] = a_2, [a_2, a_1] = b^{9s}, [a_2, b] = a_1^{-3} a_2^{3t} \rangle$ ,  
where  $s, t = 1, 2$ . Moreover,  $|G| = 3^7$ ,  $\Phi(G) = \langle a_2, a_1^3, b^3 \rangle \cong C_3 \times C_9 \times C_9$ ,  
 $G' = \langle a_2, a_1^3, b^9 \rangle \cong C_9 \times C_3 \times C_3$  and  $Z(G) = \langle a_2^3, b^3 \rangle \cong C_3 \times C_9$ .

(Mx)  $c(G) = 4$ ,  $G$  has a unique three-generator maximal subgroup  $M$ ,  $|M'| = p$  where  $p \geq 5$  and  $G/M' \in \mathcal{A}_3$ . In this case,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1 + p)$  and  $\alpha_1(G) = 2p^2$ .

(M62)  $\langle b, a_1; a_2 \mid b^{p^2} = a_1^{p^2} = a_2^p = a_3^p = 1, [a_1, b] = a_2, [a_2, a_1] = b^{\nu p}, [a_2, b] = a_3, [a_3, b] = a_1^{t p}, [a_3, a_1] = 1 \rangle$ , where  $p \geq 5$ ,  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ ,  $t = t_1, t_2, \dots, t_{(3,p-1)}$ , where  $t_1, t_2, \dots, t_{(3,p-1)}$  are the coset representatives of the subgroup  $(F_p^*)^3$  in  $F_p^*$ . Moreover,  $|G| = p^6$ ,  $\Phi(G) = G' = \langle a_2, a_3, a_1^p, b^p \rangle \cong C_p^4$  and  $Z(G) = \langle a_1^p, b^p \rangle \cong C_p^2$ .

**Proof** If  $\Phi(G')G_3 \leq Z(G)$  and  $\Phi(G')G_3 \leq C_p^2$ , then  $G$  is one of the groups listed in [3, Theorem 3.5, 4.6, 5.5, 5.8 and 6.5]. By [3, Theorem 3.6, 4.7, 5.1-5.2, 5.5-5.6 and 6.1], we get the groups (M1)-(M47). For convenience, in Table 11 we give the correspondence from those groups (M1)-(M47) to [3, Theorem 3.5, 4.6, 5.5, 5.8 and 6.5]. In the following, we may assume that  $\Phi(G')G_3 \not\leq Z(G)$  or  $\Phi(G')G_3 \not\leq C_p^2$ .

If  $G$  has two distinct three-generator maximal subgroup  $M_1$  and  $M_2$ , then, by Lemma 2.6 (2),  $M'_1 \leq C_p^3$  and  $M'_2 \leq C_p^3$ . By hypothesis,  $M'_1 \leq Z(G)$  and  $M'_2 \leq Z(G)$ . Let  $N = M'_1 M'_2$ . Then  $\exp(N) = p$  and  $N \leq Z(G)$ . Let  $\bar{G} = G/N$ . Then  $\bar{G}$  has two distinct abelian subgroups  $\bar{M}_1$  and  $\bar{M}_2$  of index  $p$ . By Lemma 2.2 we have  $\bar{G} \in \mathcal{A}_1$ . It follows that  $\Phi(G')G_3 = N$ . By above assumption,  $|N| \geq p^3$ . Hence  $G$  is not metacyclic. By Lemma 2.1,  $\bar{G}$  is not metacyclic. Let

$$\bar{G} = \langle \bar{a}, \bar{b}; \bar{c} \mid \bar{a}^{p^n} = \bar{b}^{p^m} = \bar{c}^p = 1, [\bar{a}, \bar{b}] = \bar{c}, [\bar{c}, \bar{a}] = [\bar{c}, \bar{b}] = 1 \rangle,$$

Groups	Groups in [3, Theorem 3.5, 4.6, 5.5, 5.8 and 6.5]	Groups	Groups in [3, Theorem 3.5, 4.6, 5.5, 5.8 and 6.5]
(M1)	(C1)	(M25)	(L2) where $n = 2$ and $t^2 - 4t \notin F_p^2$
(M2)	(C2)	(M26)	(L4) where $n = 2$ and $-\nu \notin F_p^2$
(M3)	(D1) where $p > 2 = n$	(M27)	(L5) where $n = 2$ and $-\nu \notin F_p^2$
(M4)	(D2) where $p > 2 = n$	(M28)	(L6) where $n = 2$
(M5)	(D3)	(M29)	(L8) where $n = 2$
(M6)	(I4) where $p > 2 = n$	(M30)	(M1) where $m = 2$ and $s = 0$
(M7)	(D5)	(M31)	(M1) where $m = 2$ , $s = 0$ and $\nu = \eta$
(M8)	(E2) where $p > 2 = m$	(M32)	(M2) where $m = 2$
(M9)	(E5) where $p > 2 = m$	(M33)	(M4) where $m = 2$
(M10)	(E8) where $p > 2 = m$	(M34)	(M5) where $m = 2$
(M11)	(E3)	(M35)	(M6) where $m = 2$ and $t^2 + 4t \notin F_p^2$
(M12)	(E6)	(M36)	(M8) where $m = 2$ and $\nu = \eta$
(M13)	(E9)	(M37)	(N3)
(M14)	(G1) where $p = m = 2$	(M38)	(N8)
(M15)	(G2) where $p = m = 2$	(M39)	(N9)
(M16)	(G3) where $p = m = 2$	(M40)	(P1) where $n = 1$
(M17)	(J2) where $p = m = 2$	(M41)	(P3) where $n = 2$ and $-\nu \notin F_p^2$
(M18)	(J4) where $p = m = 2$	(M42)	(p4) where $n = 2$ and $-r \notin F_p^2$
(M19)	(J6) where $p = m = 2$	(M43)	(P10) where $n = 1$
(M20)	(K2)	(M44)	(Q1) where $n = 2$ and $(st\nu)^2 - 4\nu(s+1) \notin F_p^2$
(M21)	(K4)	(M45)	(Q1) where $n = 2$ , $s = -1$ and $t = 0$
(M22)	(K8)	(M46)	(Q2) where $n = 2$
(M23)	(K10)	(M47)	(S4) where $n = 2$
(M24)	(L1) where $n = 2$ and $1 + 4s \notin F_p^2$		

Table 11: The correspondence from Theorem 5.13 to [3, Theorem 3.5, 4.6, 5.5, 5.8 and 6.5]

where  $n \geq m$ . Then  $N = \langle c^p, [c, a], [c, b] \rangle$ ,  $|N| = p^3$  and  $|G| = p^{n+m+4}$ . Since  $c^p \neq 1$ , by calculation we have  $[a, b^p] \neq 1$ . It follows that  $m \geq 2$ . Since  $\langle c, b \rangle \in \mathcal{A}_1$ , we have  $|\langle c, b \rangle| = p^{n+m+2}$ . It follows that  $n \leq 2$  and hence  $n = m = 2$ . Moreover,

$$N = \langle c^p, a^{p^2}, b^{p^2} \rangle, \text{ where } [c, a] \notin \langle c^p, a^{p^2} \rangle \text{ and } [c, b] \notin \langle c^p, b^{p^2} \rangle.$$

By suitable replacement, we may assume that  $[c, a] = b^{rp^2} c^{sp}$  and  $[c, b] = a^{up^2} b^{vp^2} c^{wp}$ , where  $r, s, u, v, w \in F_p$ ,  $r \neq 0$  and  $u \neq 0$ . If  $p = 2$ , then we can prove that  $G \in \mathcal{A}_4$ . The details are omitted. Hence  $p \geq 3$ . If  $s^2 - 4r$  is a square, then the equation  $x^2 - sx + r = 0$  has a solution  $x_1$ . By calculation we have  $\langle a, cb^{x_1 p} \rangle$  is not abelian and of order  $p^5$ , which contradicts that  $G \in \mathcal{A}_3$ . Hence  $s^2 - 4r$  is not a square. By Lemma 2.10, the equation  $x^2 + sxy + ry^2 + wx + vy - u = 0$  has a solution  $(x_0, y_0)$ . By calculation,  $\langle ca^{x_0 p}, ba^{y_0} \rangle$  is not abelian and of order  $p^5$ , which contradicts that  $G \in \mathcal{A}_3$  again.

By above argument,  $G$  has a unique three-generator maximal subgroup  $M$ . Let  $\bar{G} = G/M'$ . Then  $\bar{G}$  has a three-generator abelian subgroup  $M/M'$  of index  $p$ , and every non-abelian subgroup of  $\bar{G}$  is generated by two elements. By Lemma 2.12 (2),  $p \geq 3$ . Since  $\bar{G} \in \mathcal{A}_2$  or  $\mathcal{A}_3$ ,  $\bar{G}$  is either one of the groups (4)–(7) in Lemma 2.5 or, by Theorem 5.1, one of the groups (F4)–(F8).

We claim that  $G'$  is abelian. Otherwise,  $|G : G'| = p^2$  and  $G'$  is minimal non-abelian. By Lemma 2.14,  $G'$  is abelian, a contradiction.

Case 1:  $\bar{G}$  is the group (4) in Lemma 2.5. That is,  $\bar{G} = \langle \bar{a}_1, \bar{b}; \bar{a}_2, \bar{a}_3 \mid \bar{a}_1^p = \bar{a}_2^p = \bar{a}_3^p = \bar{b}^{p^m} = 1, [\bar{a}_1, \bar{b}] = \bar{a}_2, [\bar{a}_2, \bar{b}] = \bar{a}_3, [\bar{a}_3, \bar{b}] = 1, [\bar{a}_i, \bar{a}_j] = 1 \rangle$ , where  $1 \leq i, j \leq 3$ .

Since  $d(\bar{M}) = 3$ , we have  $m = 1$  and  $M = \langle a_1, a_2, a_3 \rangle$ . Hence  $p \geq 5$ . Since  $G'$  is abelian,  $[a_2, a_3] = 1$ . By calculation we have

$$[a_3, a_1] = [a_2, b, a_1] = [a_2, a_1, b] = 1.$$

It follows that

$$[a_1, a_2] \neq 1, |M'| = p \text{ and } |G| = p^5.$$

By calculation we have

$$a_3^p = [a_2, b]^p = [a_2^p, b] = 1 \text{ and } a_2^p = [a_1^p, b] = 1.$$

Since  $\langle a_2, b \rangle \in \mathcal{A}_2$ ,  $[a_3, b] \neq 1$  and hence  $G$  is of maximal class.

If  $\exp(G) = p^2$  and  $\exp(M) = p$ , then  $a_1^p = 1$  and  $b^{p^2} = 1$ . Hence  $G = \langle b, a_1, a_2, a_3 \rangle$  has following relations:

$$b^{p^2} = a_1^p = a_2^p = a_3^p = 1, [a_1, b] = a_2, [a_2, b] = a_3, [a_3, b] = b^{ip}, [a_2, a_1] = b^{jp}$$

where  $i, j \in F_p^*$ . By Lemma 5.8 we get groups (M48)–(M51).

If  $\exp(G) = p^2$  and  $\exp(M) = p^2$ , then  $a_1^{p^2} = 1$ . By suitable replacement, we may assume that  $b^p = 1$ . Hence  $G = \langle b, a_1, a_2, a_3 \rangle$  has following relations:

$$b^p = a_1^{p^2} = a_2^p = a_3^p = 1, [a_1, b] = a_2, [a_2, b] = a_3, [a_3, b] = a_1^{ip}, [a_2, a_1] = a_1^{jp}$$

where  $i, j \in F_p^*$ . By Lemma 5.9 we get the group (M52).

If  $\exp(G) = p$ , then we get the group (M53).

Case 2:  $\bar{G}$  is the group (5) in Lemma 2.5. That is,  $\bar{G} = \langle \bar{a}_1, \bar{b}; \bar{a}_2 \mid \bar{a}_1^p = \bar{a}_2^p = \bar{b}^{p^{m+1}} = 1, [\bar{a}_1, \bar{b}] = \bar{a}_2, [\bar{a}_2, \bar{b}] = \bar{b}^{p^m}, [\bar{a}_1, \bar{a}_2] = 1 \rangle$ .

Since  $d(\bar{M}) = 3$ ,  $M = \langle a_1, a_2, b^p \rangle$ . Let  $N = \langle a_2, b, M' \rangle$ . Then  $N$  is maximal in  $G$  and  $d(N) = 2$ . It follows that  $N \in \mathcal{A}_2$ . Since  $G'$  is abelian, we have  $[b^{p^m}, a_2] = 1$  and hence  $c(N) = 2$ . By calculation we get  $b^{p^{m+1}} = [a_2, b]^p = [a_2^p, b] = 1$  and hence  $|N'| = p$ . By Lemma 2.2 we have  $N \in \mathcal{A}_1$ , a contradiction.

Case 3:  $\bar{G}$  is the group (6) in Lemma 2.5. That is,  $\bar{G} = \langle \bar{a}_1, \bar{b}; \bar{a}_2 \mid \bar{a}_1^{p^2} = \bar{a}_2^p = \bar{b}^{p^m} = 1, [\bar{a}_1, \bar{b}] = \bar{a}_2, [\bar{a}_2, \bar{b}] = \bar{a}_1^{\nu p}, [\bar{a}_1, \bar{a}_2] = 1 \rangle$ , where  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ .

Since  $d(\bar{M}) = 3$ , we have  $m \geq 2$  and  $M = \langle a_1, a_2, b^p \rangle$ . Let  $N = \langle a_2, b, M' \rangle$ . Then  $N$  is maximal in  $G$  and  $d(N) = 2$ . It follows that  $N \in \mathcal{A}_2$ . By calculation,  $a_1^{\nu p^2} = [a_2, b]^p = [a_2^p, b] = 1$ . Hence  $a_1^{p^2} = 1$  and  $\exp(N') = p$ . If  $c(N) = 2$ , then  $G \in \mathcal{A}_1$ , a contradiction. Hence  $c(N) = 3$ . By calculation,  $[a_2, b^p] = 1$ . Since  $[a_1^p, a_2] = 1$ , we have  $[a_1^p, b] = a_2^p \neq 1$ .

If  $|M'| = p^2$ , then  $M' = \langle [a_1, a_2] \rangle \times \langle [a_1, b^p] \rangle$ . Since  $G \in \mathcal{A}_3$ , we have  $|\langle [a_1, a_2] \rangle| = |\langle [a_1, b^p] \rangle| = p^{m+3}$ . It follows that  $m = 2$ ,  $|G| = p^7$  and  $M' = \langle a_2^p, b^{p^2} \rangle$ . By suitable replacement, we may assume that

$$G = \langle b, a_1; a_2 \mid b^{p^3} = a_1^{p^2} = a_2^{p^2} = 1, [a_1, b] = a_2, [a_2, a_1] = b^{\nu_1 p^2} a_2^{rp}, [a_2, b] = a_1^{\nu_2 p} a_2^{sp} \rangle,$$

where  $\nu_1, \nu_2 = 1$  or a fixed quadratic non-residue modulo  $p$ . By Lemma 5.10 we get groups (M54)–(M57).

In the following, we may assume that  $|M'| = p$ . Hence  $M' = \langle a_2^2 \rangle$  and  $|G| = p^{m+4}$ .

If  $[a_2, a_1] \neq 1$ , then  $|\langle a_1, a_2 \rangle| = p^{m+2}$  since  $G \in \mathcal{A}_3$ . It follows that  $m = 2$  and  $|G| = p^6$ . By suitable replacement, we may assume that

$$G = \langle b, a_1; a_2 \mid a_1^{p^2} = a_2^{p^2} = 1, b^{p^2} = a_2^{tp}, [a_1, b] = a_2, [a_2, a_1] = a_2^p, [a_2, b] = a_1^{\nu p} \rangle,$$

where  $t \in F_p$ ,  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ . By Lemma 5.11 we get the group (M58).

If  $[a_2, a_1] = 1$ , then, by suitable replacement, we may assume that

$$G = \langle b, a_1; a_2 \mid a_1^{p^2} = a_2^{p^2} = 1, b^{p^m} = a_2^{tp}, [a_1, b] = a_2, [a_2, a_1] = 1, [a_2, b] = a_1^{\nu p} a_2^{sp} \rangle,$$

where  $s, t \in F_p$ . By Lemma 5.12 we get groups (M59)–(M60).

Case 4:  $\bar{G}$  is the group (7) in Lemma 2.5. That is,  $\bar{G} = \langle \bar{a}_1, \bar{b}; \bar{a}_2 \mid \bar{a}_1^9 = \bar{a}_2^3 = 1, \bar{b}^3 = \bar{a}_1^3, [\bar{a}_1, \bar{b}] = \bar{a}_2, [\bar{a}_2, \bar{b}] = \bar{a}_1^{-3}, [\bar{a}_2, \bar{a}_1] = 1 \rangle$ .

In this case, we have  $d(\bar{M}) = d(\langle \bar{a}_2, \bar{a}_1 \rangle) = 2$ , a contradiction.

Case 5:  $\bar{G}$  is the group (F4) in Main Theorem. That is,  $\bar{G} = \langle \bar{a}_1, \bar{b}; \bar{a}_2 \mid \bar{a}_1^9 = \bar{a}_2^9 = \bar{b}^{3^m} = 1, [\bar{a}_1, \bar{b}] = \bar{a}_2, [\bar{a}_2, \bar{b}] = \bar{a}_1^{-3} \bar{a}_2^{3t}, [\bar{a}_1, \bar{a}_2] = 1 \rangle$ , where  $t = 1, 2$ .

Since  $d(\bar{M}) = 3$ , we have  $m \geq 2$  and  $M = \langle a_1, a_2, b^3 \rangle$ . By Lemma 2.6 (4),  $|M'| = 3$ . By Lemma 2.6 (3),  $m = 2$  and hence  $|G| = 3^7$ . Let  $N = \langle a_2, b, M' \rangle$ . Then  $N$  is maximal in  $G$  and  $d(N) = 2$ . It follows that  $N \in \mathcal{A}_2$ . By calculation,  $a_2^9 = [a_1^3, b]^3 = [a_1^9, b] = 1$ . Since  $|\langle a_2^3, b \rangle| \leq 3^4$ , we have  $[a_2^3, b] = 1$ . By calculation,  $a_1^9 = [b, a_2]^3 = [b, a_2^3] = 1$ . Hence  $\exp(N') = 3$ . Let  $L = \langle a_1^3, b \rangle$ . Then  $L \in \mathcal{A}_1$ . It follows that  $|L| = 3^5$ . Hence  $b^9 \neq 1$ . That is,  $M' = \langle b^9 \rangle$ . By calculation,  $[a_2, b^3] = 1$  and  $[a_1, b^3] = 1$ . Hence  $[a_2, a_1] \neq 1$ . Then we may assume

$$G = \langle b, a_1; a_2 \mid a_1^9 = a_2^9 = b^{27} = 1, [a_1, b] = a_2, [a_2, a_1] = b^{9s}, [a_2, b] = a_1^{-3} a_2^{3t} b^{9r} \rangle,$$

where  $s, t = 1, 2$ . By replacing  $b$  with  $a_1^{-rs}$  we get the group (M61).

Case 6:  $\bar{G}$  is the group (F5) in Main Theorem. That is,  $\bar{G} = \langle \bar{a}_1, \bar{b}; \bar{a}_2 \mid \bar{a}_1^9 = \bar{a}_2^9 = 1, \bar{b}^{3^m} = \bar{a}_2^{-3}, [\bar{a}_1, \bar{b}] = \bar{a}_2, [\bar{a}_2, \bar{b}] = \bar{a}_1^{-3} \bar{a}_2^{-3}, [\bar{a}_1, \bar{a}_2] = 1 \rangle$ .

Since  $d(\bar{M}) = 3$ , we have  $m \geq 2$  and  $M = \langle a_1, a_2, b^3 \rangle$ . By Lemma 2.6 (4),  $|M'| = 3$ . By Lemma 2.6 (3),  $m = 2$  and hence  $|G| = 3^7$ . Let  $N = \langle a_2, b, M' \rangle$ . Then  $N$  is maximal in  $G$  and  $d(N) = 2$ . It follows that  $N \in \mathcal{A}_2$ . By calculation we have

$$a_2^9 = [a_1^3, b]^3 = [a_1^9, b] = 1 \text{ and } a_1^9 = [b, a_2]^3 = [b, a_2^3] = 1.$$

Hence  $\exp(N') = 3$ . Let  $L = \langle a_1^3, b \rangle$ . Then  $L \in \mathcal{A}_1$ . It follows that  $|L| = 3^5$ . Hence  $b^9 \neq a_2^{-3}$ . That is,  $M' = \langle a_2^3 b^9 \rangle$ . By calculation,  $[a_2, b^3] = 1$  and  $[a_1, b^3] = 1$ . Hence  $[a_2, a_1] \neq 1$ . Then we may assume  $[a_2, a_1] = a_2^{3s} b^{9s}$ , where  $s = 1, 2$ . By calculation,  $\langle a_2 b^3, a_1 \rangle \in \mathcal{A}_1$  and is of order  $3^4$ , which contradicts that  $G \in \mathcal{A}_3$ .

Case 7:  $\bar{G}$  is the group (F6) in Main Theorem. That is,  $\bar{G} = \langle \bar{a}_1, \bar{b}; \bar{a}_2, \bar{a}_3, \bar{a}_4 \mid \bar{a}_i^p = \bar{b}^{p^m} = 1, [\bar{a}_j, \bar{b}] = \bar{a}_{j+1}, [\bar{a}_4, \bar{b}] = 1, [\bar{a}_i, \bar{a}_j] = 1 \rangle$ , where  $p \geq 5$ ,  $1 \leq i \leq 4$ ,  $1 \leq j \leq 3$ .

In this case, we have  $d(\bar{M}) = d(\langle \bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{b}^p \rangle) = 5$ , a contradiction.

Case 8:  $\bar{G}$  is the group (F7) in Main Theorem. That is,  $\bar{G} = \langle \bar{a}_1, \bar{b}; \bar{a}_2, \bar{a}_3 \mid \bar{a}_i^p = \bar{b}^{p^{m+1}} = 1, [\bar{a}_j, \bar{b}] = \bar{a}_{j+1}, [\bar{a}_3, \bar{b}] = \bar{b}^{p^m}, [\bar{a}_i, \bar{a}_j] = 1 \rangle$ , where  $p \geq 5$ ,  $1 \leq i \leq 3$ ,  $1 \leq j \leq 2$ .

In this case, we have  $d(\bar{M}) = d(\langle \bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{b}^p \rangle) = 4$ , a contradiction.

Case 9:  $\bar{G}$  is the group (F8) in Main Theorem. That is,  $\bar{G} = \langle a_1, b; \bar{a}_2, \bar{a}_3 \mid \bar{a}_1^{p^2} = \bar{a}_i^p = \bar{b}^{p^m} = 1, [\bar{a}_j, b] = \bar{a}_{j+1}, [\bar{a}_3, \bar{b}] = \bar{a}_1^{tp}, [\bar{a}_i, \bar{a}_j] = 1 \rangle$ , where  $2 \leq i \leq 3$ ,  $1 \leq j \leq 2$ , and  $t = t_1, t_2, \dots, t_{(3,p-1)}$ , where  $p \geq 5$ ,  $t_1, t_2, \dots, t_{(3,p-1)}$  are the coset representatives of the subgroup  $(F_p^*)^3$  in  $F_p^*$ .

Since  $d(\bar{M}) = 3$ , we have  $m = 1$  and  $M = \langle a_1, a_2, a_3 \rangle$ . Since  $G'$  is abelian,  $[a_2, a_3] = 1$ . By calculation we have

$$[a_3, a_1] = [a_2, b, a_1] = [a_2, a_1, b] = 1.$$

It follows that

$$[a_1, a_2] \neq 1, |M'| = p \text{ and } |G| = p^6.$$

By calculation we have

$$a_3^p = [a_2, b]^p = [a_2^p, b] = 1, a_2^p = [a_1^p, b] = 1 \text{ and } a_1^{tp^2} = [a_3^p, b] = 1.$$

Let  $[a_2, a_1] = d$ . Then we may assume

$$G = \langle b, a_1; a_2, a_3, d \mid a_1^{p^2} = a_2^p = a_3^p = 1, b^p = d^s, [a_1, b] = a_2, [a_2, a_1] = d, [a_2, b] = a_3, [a_3, b] = a_1^{tp} d^r \rangle.$$

Since  $[a_3, ba_1^x] = a_1^{tp} d^r \neq 1$ , we have  $|\langle a_3, ba_1^x \rangle| = p^4$  for any  $x$ . It follows that the equation  $st - rx \equiv 0 \pmod{p}$  about  $x$  has no solution. Hence we have  $r = 0$  and  $s \neq 0$ . By suitable replacement we get the group (M62).

We calculate the  $(\mu_0, \mu_1, \mu_2)$  and  $\alpha_1(G)$  of those groups in Theorem 5.13 as follows. Since  $d(G) = 2$ ,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1 + p)$ . In the following, we calculate  $\alpha_1(G)$ .

**Case 1.**  $G$  is one of the groups (M1)–(M19).

Since  $|G'| = p^2$ ,  $|M'| = p$  for any  $M \in \Gamma_1$ . By Lemma 2.6 (7),  $\alpha_1(M) = p^2$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = (1 + p)p^2 = p^2 + p^3.$$

**Case 2.**  $G$  is one of the groups (M20)–(M36).

In this case,  $G = \langle a, b \rangle$  such that  $[a, b] = c$ ,  $c^{p^2} = 1$ ,  $[c, a] = 1$  and  $[c, b] \notin \langle c^p \rangle$ . All maximal subgroups of  $G$  are:

$$M = \langle c, a, \Phi(G) \rangle;$$

$$M_i = \langle c, ba^i, \Phi(G) \rangle, \text{ where } 0 \leq i \leq p-1.$$

It is easy to see that  $|M'| = p$ . By Lemma 2.6 (7),  $\alpha_1(M) = p^2$ . By calculation,  $d(M_i) = 3$  and  $|M'_i| = p^2$ . Hence  $\alpha_1(M_i) = p^2 + p$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = p^2 + p \times (p^2 + p) = p^3 + 2p^2.$$

**Case 3.**  $G$  is one of the groups (M37).

All maximal subgroups of  $G$  are:

$$M = \langle c, a, \Phi(G) \rangle;$$

$$M_i = \langle c, ba^i, \Phi(G) \rangle, \text{ where } i = 0, 1.$$

By calculation,  $d(M) = 3$  and  $|M'| = p^2$ . Hence  $\alpha_1(M) = p^2 + p$ . Similarly,  $\alpha_1(M_i) = p^2 + p$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = (1 + p)(p^2 + p) = p^3 + 2p^2 + p.$$

**Case 4.**  $G$  is one of the groups (M38)–(M39).

In this case,  $G = \langle a, b \rangle$  such that  $[a, b] = c, c^4 = 1, [c, a] = c^2$  and  $[c, b] \notin \langle c^2 \rangle$ . All maximal subgroups of  $G$  are:

$$M = \langle c, a, \Phi(G) \rangle;$$

$$M_i = \langle c, ba^i, \Phi(G) \rangle, \text{ where } i = 0, 1.$$

By calculation,  $M = \langle c, a, b^2 \rangle$  such that  $d(M) = 3$  and  $|M'| = p^2 = 4$ ,  $M_i = \langle c, ba^i, a^2 \rangle$  such that  $d(M_i) = 3$  and  $|M'_i| = p = 2$ . By Lemma 2.6,  $\alpha_1(M) = p^2 + p$  and  $\alpha_1(M_i) = p^2$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = (p^2 + p) + p \times p^2 = p^3 + p^2 + p.$$

**Case 5.**  $G$  is one of the groups (M40)–(M43).

Let  $H \in \Gamma_1$ . Then  $|H'| = p$ . By Lemma 2.6,  $\alpha_1(H) = p^2$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = (1 + p) \times p^2 = p^3 + p^2.$$

**Case 6.**  $G$  is one of the groups (M44)–(M46).

In this case,  $G = \langle a, b \rangle$  such that  $[a, b] = c, c^{p^2} = 1, [c, a] = c^p$  and  $[c, b] \notin \langle c^p \rangle$  where  $p > 2$ . All maximal subgroups of  $G$  are:

$$M = \langle c, a, \Phi(G) \rangle;$$

$$M_i = \langle c, ba^i, \Phi(G) \rangle, \text{ where } i = 0, 1.$$

By calculation,  $M = \langle c, a, b^2 \rangle$  such that  $d(M) = 3$  and  $|M'| = p$ ,  $M_i = \langle c, ba^i, a^2 \rangle$  such that  $d(M_i) = 3$  and  $|M'_i| = p^2$ . By Lemma 2.6,  $\alpha_1(M) = p^2$  and  $\alpha_1(M_i) = p^2 + p$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = p^2 + p \times (p^2 + p) = p^3 + 2p^2.$$

**Case 7.**  $G$  is the group (M47).

Let  $H \in \Gamma_1$ . Then  $|H'| = p$ . By Lemma 2.6,  $\alpha_1(H) = p^2$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = (1 + p) \times p^2 = p^3 + p^2.$$

**Case 8.**  $G$  is one of the groups (M48)–(M53).

In this case,  $\Phi(G)$  is abelian and  $|M'| = p$ . Let  $H \in \Gamma_1 \setminus \{M\}$ . Then  $d(H) = 2$ . Hence  $\alpha_1(H) = p$  and  $\alpha_1(M) = p^2$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = p^2 + p \times p = 2p^2.$$

**Case 9.**  $G$  is one of the groups (M54)–(M57).

In this case,  $\Phi(G)$  is abelian and  $|M'| = p^2$ . Let  $H \in \Gamma_1 \setminus \{M\}$ . Then  $d(H) = 2$ . Hence  $\alpha_1(H) = p$  and  $\alpha_1(M) = p^2 + p$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = (p^2 + p) + p \times p = 2p^2 + p.$$

**Case 10.**  $G$  is one of the groups (M58)–(M62).

In this case,  $\Phi(G)$  is abelian and  $|M'| = p$ . Let  $H \in \Gamma_1 \setminus \{M\}$ . Then  $d(H) = 2$ . Hence  $\alpha_1(H) = p$  and  $\alpha_1(M) = p^2$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = p^2 + p \times p = 2p^2.$$

□

**Lemma 5.14.** *Suppose that  $p \geq 3$  and  $G(\nu, k) = \langle a, b, x \mid a^{p^3} = b^{p^3} = x^p = 1, [a, b] = a^p, [x, b] = a^{p^2}, [x, a] = b^{\nu p^2} a^{kp^2} \rangle$ , where  $k \in F_p$  and  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ .*

- (1) *If  $G = G(\nu, k)$ , then  $|G| = p^7$ ,  $G' = \langle a^p, b^{p^2} \rangle$ ,  $G_3 = \langle a^{p^2} \rangle$ ;*
- (2)  *$G(\nu', k') \cong G(\nu, k)$  if and only if  $\nu' = \nu$  and  $k' = \pm k$ .*
- (3)  *$G(\nu, k) \in A_3$  if and only if  $k^2 + 4\nu$  is not a square.*



**Proof** (1) Let

$$K = \langle a, b \mid a^{p^3} = b^{p^3} = 1, [a, b] = a^p \rangle.$$

We define an automorphism  $\gamma$  of  $K$  as follows:

$$a^\gamma = aa^{-kp^2}b^{-\nu p^2}, \quad b^\gamma = ba^{-p^2}.$$

Then

$$a^{\gamma^p} = 1, \quad b^{\gamma^p} = 1, \quad o(\gamma) = p.$$

It follows from the cyclic extension theory that  $G = \langle K, x \rangle$  is a cyclic extension of  $K$  by  $C_p$ . Hence  $|G| = p^7$ . It is easy to check that  $G' = \langle a^p, b^{p^2} \rangle$ ,  $G_3 = \langle a^{p^2} \rangle$ .

(2) For convenience, let  $G = \langle a, b, x \rangle \cong G(\nu, k)$ ,  $\bar{G} = \langle \bar{a}, \bar{b}, \bar{x} \rangle \cong G(\nu', k')$  and  $\theta$  be an isomorphism from  $\bar{G}$  to  $G$ . Since  $M = \langle x, b, a^p \rangle$  is the unique three-generator maximal subgroups such that  $M' \cong C_p^2$ ,  $M \text{ char } G$ . Similarly,  $\bar{M} = \langle \bar{x}, \bar{b}, \bar{a}^p \rangle \text{ char } \bar{G}$ . Since  $M, \Phi(G), \Omega_1(G) = \langle x, a^{p^2}, b^{p^2} \rangle$  and  $\bar{M}, \Phi(\bar{G}), \Omega_1(\bar{G}) = \langle \bar{x}, \bar{a}^{p^2}, \bar{b}^{p^2} \rangle$  are characteristic in  $G$  and  $\bar{G}$  respectively, we may assume that  $\bar{a}^\theta = a^l x^n w$ ,  $\bar{b}^\theta = b^i x^j y$ ,  $\bar{x}^\theta = x^r z$  where  $l, i, r \in F_p^*$  and  $w, y \in \Phi(G)$ ,  $z \in \Omega_1(M)$ .

Since  $[a^l, b^{ip}] = [\bar{a}^\theta, (\bar{b}^p)^\theta] = [\bar{a}, \bar{b}^p]^\theta = (\bar{a}^{p^2})^\theta$ , we have  $a^{lip^2} = a^{lp^2}$ . Comparing index of  $a^{p^2}$  in two sides, we have  $i = 1$ .

Since  $[x^r, b] = [\bar{x}^\theta, \bar{b}^\theta] = (\bar{a}^{p^2})^\theta = a^{lp^2}$ , we have  $a^{rp^2} = a^{lp^2}$ . Comparing index of  $a^{p^2}$  in two sides, we have  $r = l$ .

Since  $[x^r, a^r] = [\bar{x}^\theta, \bar{a}^\theta] = (\bar{b}^{\nu' p^2} \bar{a}^{k' p^2})^\theta = b^{\nu' p^2} a^{rk' p^2}$ , we have  $b^{r^2 \nu p^2} a^{r^2 k p^2} = b^{\nu' p^2} a^{rk' p^2}$ . Comparing indexes of  $a^{p^2}$  and  $b^{p^2}$  in two sides, we have  $\nu' = r^2 \nu$  and  $k' = rk$ .

Since  $\nu' = r^2 \nu$ ,  $\nu = \nu' = 1$  or  $\nu' = \nu$  is a fixed quadratic non-residue modula  $p$ . Moreover,  $r^2 = 1$  and hence  $r = \pm 1$ . It follows that  $k' = \pm k$ .

On the other hand, if  $\nu' = \nu$  and  $k' = -k$ , then,  $\theta : \bar{a} \rightarrow a^{-1}, \bar{b} \rightarrow b, \bar{x} \rightarrow x^{-1}$  is an isomorphism from  $\bar{G}$  to  $G$ .

(3) All maximal subgroups of  $G(\nu, k)$  are:

$$N = \langle b, x, a^p \rangle;$$

$$N_i = \langle ab^i, x, b^p \rangle \text{ where } 0 \leq i, j \leq p-1.$$

$$N_{ij} = \langle ax^i, bx^j \rangle \text{ where } 0 \leq i, j \leq p-1.$$

It is easy to see that  $N = \langle x, b \rangle * \langle a^p \rangle \in \mathcal{A}_2$ . Since  $N_{ij} \cong \langle a, b \rangle$ ,  $N_{ij} \in \mathcal{A}_2$ . Hence  $G(\nu, k) \in \mathcal{A}_3$  if and only if  $N_i \in \mathcal{A}_2$  for any  $i$ . Let  $a_1 = ab^i$  and  $b_1 = b^p$ . Then  $N_i = \langle a_1, b_1, x \rangle$  such that

$$a_1^{p^3} = b_1^{p^2} = x^p = 1, [a_1, b_1] = a_1^{p^2} b_1^{-ip}, [x, a_1] = a_1^{(k+i)p^2} b_1^{(\nu-ik-i^2)p}, [x, b_1] = 1 \rangle.$$

By calculation,  $[x, a_1] \neq 1$  for any  $i$ . Hence  $\langle x, a_1 \rangle \in \mathcal{A}_1$ . Since  $G(\nu, k) \in \mathcal{A}_3$ ,  $|\langle x, a_1 \rangle| = p^5$ . It follows that  $\nu - ik - i^2 \neq 0$  for any  $i$ . That is, equation  $i^2 + ik - \nu = 0$  about  $i$  has no solution. Hence  $k^2 + 4\nu$  is not a square.

On the other hand, if  $k^2 + 4\nu$  is not a square, then it can be proved that  $N_i \in \mathcal{A}_2$  for any  $i$ . Hence  $G(\nu, k) \in \mathcal{A}_3$  if and only if  $k^2 + 4\nu$  is not a square.  $\square$

**Theorem 5.15.** *Suppose that  $G$  is an  $\mathcal{A}_3$ -group all of whose maximal subgroups are  $\mathcal{A}_2$ -groups and there exists a three-generator maximal subgroup in  $G$ , and  $M' \leq Z(G)$  for every three-generator maximal subgroup  $M$ . Then  $d(G) = 3$  if and only if  $G$  is isomorphic to one of the following pairwise non-isomorphic groups:*

(Ni)  $\Phi(G) \leq Z(G)$  and  $c(G) = 2$ . In this case,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1 + p + p^2)$  and  $\alpha_1(G) = p^4 + p^3 + p^2$ .

(N1)  $\langle a, b, c \mid a^{p^3} = b^{p^2} = c^{p^2} = 1, [b, c] = a^{p^2}, [c, a] = c^p, [a, b] = b^{-p} \rangle$ , where  $p$  is odd; moreover,  $|G| = p^7$ ,  $G' = \langle a^{p^2}, b^p, c^p \rangle \cong C_p^3$  and  $\Phi(G) = Z(G) = \langle a^p, b^p, c^p \rangle \cong C_{p^2} \times C_p \times C_p$ .

(N2)  $\langle a, b, c, d \mid a^{p^2} = b^{p^2} = c^{p^2} = d^p = 1, [b, c] = d, [c, a] = b^p, [a, b] = c^{\nu p}, [d, a] = [d, b] = [d, c] = 1 \rangle$ , where  $p$  is odd,  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$  such that  $-\nu \notin (F_p^*)^2$ ; moreover,  $|G| = p^7$ ,  $\Phi(G) = Z(G) = \langle a^p, b^p, c^p, d \rangle \cong C_p^4$  and  $G' = \langle b^p, c^p, d \rangle \cong C_p^3$ .

(N3)  $\langle a, b, c, d \mid a^{p^2} = b^{p^2} = c^{p^2} = d^p = 1, [b, c] = d, [c, a]^{1+r} = b^{rp} c^{-p}, [a, b]^{1+r} = b^p c^p, [d, a] = [d, b] = [d, c] = 1 \rangle$ , where  $p$  is odd,  $-r \in F_p$  is not a square; moreover,  $|G| = p^7$ ,  $\Phi(G) = Z(G) = \langle a^p, b^p, c^p, d \rangle \cong C_p^4$  and  $G' = \langle b^p, c^p, d \rangle \cong C_p^3$ .

(N4)  $\langle a, b, c \mid a^8 = b^4 = c^4 = 1, [b, c] = a^4, [c, a] = c^2, [a, b] = b^2 \rangle$ ; where  $|G| = 2^7$ ,  $G' = \langle a^4, b^2, c^2 \rangle \cong C_2^3$  and  $\Phi(G) = Z(G) = \langle a^2, b^2, c^2 \rangle \cong C_4 \times C_2 \times C_2$ .

(N5)  $\langle a, b, c, d \mid a^4 = b^4 = c^4 = d^2 = 1, [b, c] = d, [c, a] = b^2, [a, b] = b^2 c^2, [d, a] = [d, b] = [d, c] = 1 \rangle$ , where  $|G| = 2^7$ ,  $\Phi(G) = Z(G) = \langle a^2, b^2, c^2, d \rangle \cong C_2^4$  and  $G' = \langle b^2, c^2, d \rangle \cong C_2^3$ .

(Nii)  $Z(G) \not\leq \Phi(G)$  and  $c(G) = 3$ . In this case,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1 + p + p^2)$  and  $\alpha_1(G) = p^3 + p^2$ .

(N6)  $\langle a, b, x \mid a^{p^{r+2}} = x^p = 1, b^{p^{r+s+t}} = a^{p^{r+s}}, [a, b] = a^{p^r}, [x, a] = [x, b] = 1 \rangle = \langle a, b \rangle \times \langle x \rangle$ , where  $r \geq 2$  for  $p = 2$ ,  $r \geq 1$  for  $p \geq 3$ ,  $t \geq 0$ ,  $0 \leq s \leq 2$  and  $r + s \geq 2$ ; moreover,  $|G| = p^{2r+s+t+3}$ ,  $\Phi(G) = \langle a^p, b^p \rangle \cong C_{p^{r+t+1}} \times C_{p^{r+s-1}}$ ,  $G' = \langle a^{p^r} \rangle \cong C_{p^2}$  and  $Z(G) = \langle a^{p^2}, b^{p^2}, x \rangle \cong C_{p^{r+t}} \times C_{p^{r+s-2}} \times C_p$ .

(N7)  $\langle a, b, x \mid a^{p^3} = b^{p^{t+3}} = 1, x^p = a^{p^2}, [a, b] = a^p, [x, a] = [x, b] = 1 \rangle$ , where  $p \geq 3$  and  $t \geq 0$ ; moreover,  $|G| = p^{t+7}$ ,  $\Phi(G) = \langle a^p, b^p \rangle \cong C_{p^{t+2}} \times C_{p^2}$ ,  $G' = \langle a^p \rangle \cong C_{p^2}$ ,  $Z(G) = \langle a^{p^2}, b^{p^2}, x \rangle \cong C_{p^{t+1}} \times C_p^2$ .

(N8)  $\langle a, b, x \mid a^{p^3} = 1, b^{p^2} = x^p = a^{p^2}, [a, b] = a^p, [x, a] = [x, b] = 1 \rangle$ , where  $p \geq 3$ ; moreover,  $|G| = p^6$ ,  $\Phi(G) = \langle a^p, b^p \rangle \cong C_{p^2} \times C_p$ ,  $G' = \langle a^p \rangle \cong C_{p^2}$  and  $Z(G) = \langle x \rangle \cong C_{p^2}$ .

(N9)  $\langle a, b, x, c \mid a^{p^2} = b^{p^2} = c^p = x^p = 1, [a, b] = c, [c, a] = b^{\nu p}, [c, b] = a^p, [x, a] = [x, b] = 1 \rangle = \langle a, b \rangle \times \langle x \rangle$ , where  $p \geq 5$ ,  $\nu$  is a fixed quadratic non-residue

modulo  $p$ ; moreover,  $|G| = p^6$ ,  $\Phi(G) = G' = \langle a^p, b^p, c \rangle \cong C_p^3$  and  $Z(G) = \langle a^p, b^p, x \rangle \cong C_p^3$ .

$$(N10) \quad \langle a, b, x; c \mid a^{p^2} = b^{p^2} = c^p = x^p = 1, [a, b] = c, [c, a] = a^{-p}b^{-lp}, [c, b] = a^{-p}, [x, a] = [x, b] = 1 \rangle = \langle a, b \rangle \times \langle x \rangle, \text{ where } p \geq 5, 4l = \rho^{2r+1} - 1, r = 1, 2, \dots, \frac{1}{2}(p-1), \rho \text{ is the smallest positive integer which is a primitive root modulo } p; \text{ moreover, } |G| = p^6, \Phi(G) = G' = \langle a^p, b^p, c \rangle \cong C_p^3 \text{ and } Z(G) = \langle a^p, b^p, x \rangle \cong C_p^3.$$

$$(N11) \quad \langle a, b, x; c \mid a^9 = b^9 = c^3 = x^3 = 1, [a, b] = c, [c, a] = b^{-3}, [c, b] = a^3, [a^3, b] = [x, a] = [x, b] = 1 \rangle = \langle a, b \rangle \times \langle x \rangle; \text{ moreover, } |G| = 3^6, \Phi(G) = G' = \langle a^3, b^3, c \rangle \cong C_3^3 \text{ and } Z(G) = \langle a^3, b^3, x \rangle \cong C_3^3.$$

$$(N12) \quad \langle a, b, x; c \mid a^9 = b^9 = c^3 = x^3 = 1, [a, b] = c, [c, a] = b^{-3}, [c, b] = a^{-3}, [x, a] = [x, b] = 1 \rangle = \langle a, b \rangle \times \langle x \rangle; \text{ moreover, } |G| = 3^6, \Phi(G) = G' = \langle a^3, b^3, c \rangle \cong C_3^3 \text{ and } Z(G) = \langle a^3, b^3, x \rangle \cong C_3^3.$$

(Niii)  $Z(G) < \Phi(G)$  and  $G$  has at least two three-generator maximal subgroups, in this case,  $c(G) = 2$  for (N24) and  $c(G) = 3$  for else. Moreover,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1 + p + p^2)$  and  $\alpha_1(G) = p^3 + p^2$  except for (N14), (N18) and (N22)–(N24).

$$(N13) \quad \langle a, b, x \mid a^8 = b^4 = x^2 = 1, [a, b] = a^{-2}, [x, a] = a^4, [x, b] = 1 \rangle; \text{ moreover, } |G| = 2^6, \Phi(G) = \langle a^2, b^2 \rangle \cong C_4 \times C_2, G' = \langle a^2 \rangle \cong C_4, Z(G) = \langle a^4, b^2 \rangle \cong C_2^2, (\mu_0, \mu_1, \mu_2) = (0, 0, 7) \text{ and } \alpha_1(G) = 12;$$

$$(N14) \quad \langle a, b, x \mid a^8 = b^8 = x^2 = 1, [a, b] = a^{-2}, [x, a] = b^4, [x, b] = a^4 \rangle; \text{ moreover, } |G| = 2^7, \Phi(G) = \langle a^2, b^2 \rangle \cong C_4 \times C_4, G' = \langle a^2, b^4 \rangle \cong C_4 \times C_2, Z(G) = \langle a^4, b^2 \rangle \cong C_4 \times C_2 \text{ and } \alpha_1(G) = 14;$$

$$(N15) \quad \langle a, b, x \mid a^8 = x^2 = 1, b^4 = a^4, [a, b] = a^{-2}, [x, a] = a^4, [x, b] = 1 \rangle; \text{ where } |G| = 2^6, \Phi(G) = \langle a^2, b^2 \rangle \cong C_4 \times C_2, G' = \langle a^2 \rangle \cong C_4 \text{ and } Z(G) = \langle b^2 \rangle \cong C_4.$$

$$(N16) \quad \langle a_1, b, x; a_2, a_3 \mid a_1^p = a_2^p = a_3^p = b^p = x^p = 1, [a_1, b] = a_2, [a_2, b] = [x, a_1] = a_3, [a_3, b] = 1, [x, b] = [a_i, a_j] = 1 \rangle, \text{ where } p > 3 \text{ and } 1 \leq i, j \leq 3; \text{ where } |G| = p^5, \Phi(G) = G' = \langle a_2, a_3 \rangle \cong C_p^2 \text{ and } Z(G) = \langle a_3 \rangle \cong C_p.$$

$$(N17) \quad \langle a_1, b, x; a_2 \mid a_1^p = a_2^p = b^p = x^{p^2} = 1, [a_1, b] = a_2, [a_2, b] = [x, a_1] = x^p, [a_2, a_1] = [a_2, x] = [x, b] = 1 \rangle, \text{ where } p > 2; \text{ moreover, } |G| = p^5, \Phi(G) = G' = \langle a_2, x^p \rangle \cong C_p^2 \text{ and } Z(G) = \langle x^p \rangle \cong C_p.$$

$$(N18) \quad \langle a_1, b, x; a_2 \mid a_1^p = a_2^p = b^{p^2} = x^{p^2} = 1, [a_1, b] = a_2, [a_2, b] = x^p, [x, a_1] = b^p, [a_2, a_1] = [a_2, x] = [x, b] = 1 \rangle, \text{ where } p > 2; \text{ moreover, } |G| = p^6, \Phi(G) = G' = \langle a_2, x^p, b^p \rangle \cong C_p^3, Z(G) = \langle b^p, x^p \rangle \cong C_p \times C_p \text{ and } \alpha_1(G) = p^2 + 2p^2 - p.$$

$$(N19) \quad \langle a_1, b, x; a_2 \mid a_1^p = a_2^p = b^{p^2} = x^p = 1, [a_1, b] = a_2, [a_2, b] = [x, a_1] = b^p, [a_2, a_1] = [a_2, x] = [x, b] = 1 \rangle, \text{ where } p > 2; \text{ moreover, } |G| = p^5, \Phi(G) = G' = \langle a_2, b^p \rangle \cong C_p^2 \text{ and } Z(G) = \langle b^p \rangle \cong C_p.$$

(N20)  $\langle a_1, b, x; a_2 \mid a_1^{p^2} = a_2^p = b^p = x^p = 1, [a_1, b] = a_2, [a_2, b] = a_1^{\nu p}, [x, a_1] = a_1^p, [a_2, a_1] = [a_2, x] = [x, b] = 1 \rangle$ , where  $p > 2$  and  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ ; moreover,  $|G| = p^5$ ,  $\Phi(G) = G' = \langle a_2, a_1^p \rangle \cong C_p^2$  and  $Z(G) = \langle a_1^p \rangle \cong C_p$ .

(N21)  $\langle a_1, b, x; a_2 \mid a_1^9 = a_2^3 = x^3 = 1, b^3 = a_1^3, [a_1, b] = a_2, [a_2, b] = a_1^{-3}, [x, a_1] = a_1^3, [a_2, a_1] = [a_2, x] = [x, b] = 1 \rangle$ ; moreover,  $|G| = 3^5$ ,  $\Phi(G) = G' = \langle a_2, a_1^3 \rangle \cong C_3^2$  and  $Z(G) = \langle a_1^3 \rangle \cong C_3$ .

(N22)  $\langle a, b, x \mid a^{p^3} = b^{p^3} = x^p = 1, [a, b] = a^p, [x, b] = a^{p^2}, [x, a] = b^{\nu p^2} a^{kp^2} \rangle$ , where  $p > 2$ ,  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ ,  $0 \leq k \leq \frac{p-1}{2}$  such that  $k^2 + 4\nu$  is not a square; moreover,  $|G| = p^7$ ,  $\Phi(G) = \langle a^p, b^p \rangle \cong C_{p^2} \times C_{p^2}$ ,  $G' = \langle a^p, b^{p^2} \rangle \cong C_{p^2} \times C_p$ ,  $Z(G) = \langle a^{p^2}, b^{p^2} \rangle \cong C_p^2$ ,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1 + p + p^2)$  and  $\alpha_1(G) = p^3 + 2p^2$ .

(N23)  $\langle a, b, x \mid a^{p^3} = b^{p^{t+3}} = x^p = 1, [a, b] = a^p, [x, b] = a^{p^2} b^{p^{t+2}}, [x, a] = 1 \rangle$ , where  $p > 2$  and  $t \geq 1$ ; moreover,  $|G| = p^{t+7}$ ,  $\Phi(G) = \langle a^p, b^p \rangle \cong C_{p^2} \times C_{p^{t+2}}$ ,  $G' = \langle a^p, b^{p^{t+2}} \rangle \cong C_{p^2} \times C_p$ ,  $Z(G) = \langle a^{p^2}, b^{p^2} \rangle \cong C_p \times C_{p^{t+1}}$ ,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1 + p + p^2)$  and  $\alpha_1(G) = p^3 + 2p^2$ .

(N24)  $\langle a, b, x \mid a^{p^{t+4}} = b^{p^3} = x^p = 1, [a, b] = a^{p^{t+2}}, [x, a] = b^{p^2}, [x, b] = 1 \rangle$ , where  $t \geq 0$ ; moreover,  $|G| = p^{t+8}$ ,  $\Phi(G) = \langle a^p, b^p \rangle \cong C_{p^2} \times C_{p^{t+3}}$ ,  $G' = \langle a^{p^{t+2}}, b^{p^2} \rangle \cong C_{p^2} \times C_p$ ,  $Z(G) = \langle a^{p^2}, b^{p^2} \rangle \cong C_p \times C_{p^{t+2}}$ ,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1 + p + p^2)$  and  $\alpha_1(G) = p^3 + 2p^2$ .

(Niv)  $Z(G) < \Phi(G)$ ,  $c(G) = 3$  and  $G$  has a unique three-generator maximal subgroup. In this case,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1 + p + p^2)$  and  $\alpha_1(G) = 2p^2$ .

(N25)  $\langle a, b, d \mid a^{p^{m+1}} = b^{p^2} = d^p = 1, [a, b] = a^{p^{m-1}}, [d, a] = b^p, [d, b] = 1 \rangle$ , where  $p > 2$  and  $m \geq 2$ ; moreover,  $|G| = p^{m+4}$ ,  $\Phi(G) = \langle a^p, b^p \rangle \cong C_{p^m} \times C_p$ ,  $G' = \langle a^{p^{m-1}}, b^p \rangle \cong C_{p^2} \times C_p$  and  $Z(G) = \langle a^{p^2} \rangle \cong C_{p^{m-1}}$ .

(N26)  $\langle a, b, d \mid a^{p^3} = b^{p^2} = d^p = 1, [a, b] = a^p, [d, a] = b^p, [d, b] = a^{\nu p^2} \rangle$ , where  $p > 2$ ,  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ ; moreover,  $|G| = p^6$ ,  $\Phi(G) = G' = \langle a^p, b^p \rangle \cong C_p \times C_{p^2}$  and  $Z(G) = \langle a^{p^2} \rangle \cong C_p$ .

**Proof** If  $\Phi(G) \leq Z(G)$ , then, since  $G$  has no abelian maximal subgroup,  $\Phi(G) = Z(G)$ . Hence  $G$  is one of the groups listed in [23, Theorem 3.1, 4.1, 5.1, 6.1, 7.1-7.2, 7.6]. By hypothesis, the minimal index of  $\mathcal{A}_1$ -subgroups and the maximal index of  $\mathcal{A}_1$ -subgroups are 2. By checking [23, Theorem 3.3, 4.3, 5.2, 6.3, 7.4-7.5, 7.7], we get groups (N1)–(N5).

In the following, we may assume  $\Phi(G) \not\leq Z(G)$ .

**Case 1:**  $Z(G) \not\leq \Phi(G)$ .

Then there exists a maximal subgroup  $M$  such that  $Z(G) \not\leq M$ . Let  $x \in Z(G) \setminus M$ . Then  $G = \langle M, x \rangle$  and  $x^p \in Z(M)$ . If  $d(M) = 3$ , then  $G' = M' \leq Z(G)$  and  $\exp(G') =$

Groups	Groups in [23, Theorem 4.1 & 7.1]	Groups	Groups in [23, Theorem 4.1 & 7.1]
(N1)	(D1) where $m_1 = 2$	(N4)	(M1) where $m_1 = 2$
(N2)	(E3) where $m_1 = 2$ and $-\nu \notin F_p^2$	(N5)	(N3) where $m_1 = 2$
(N3)	(E4) where $m_1 = 2$ and $-\nu \notin F_p^2$		

Table 12: The correspondence from (N1)–(N5) to [23, Theorem 4.1 & 7.1]

$p$ . It follows that  $\Phi(G) \leq Z(G)$ , a contradiction. Hence  $d(M) = 2$ . If  $M$  has an abelian subgroup of index  $p$ , then  $G$  also has an abelian subgroup of index  $p$ , a contradiction. Hence  $\alpha_1(M) = 1 + p$ .

Since any maximal subgroup of  $M$  is an  $\mathcal{A}_1$ -group,  $M$  is one of the groups (17)–(21) in Lemma 2.5.

Subcase 1.1:  $M$  is the group of Type (17) in Lemma 2.5. That is,  $M = \langle a, b \mid a^{p^{r+2}} = 1, b^{p^{r+s+t}} = a^{p^{r+s}}, [a, b] = a^{p^r} \rangle$ , where  $r \geq 2$  for  $p = 2$ ,  $r \geq 1$  for  $p \geq 3$ ,  $t \geq 0$ ,  $0 \leq s \leq 2$  and  $r + s \geq 2$ .

If  $s = 2$ , then

$$Z(M) = \langle a^{p^2} \rangle \times \langle b^{p^2} \rangle, \quad |M| = p^{2r+t+4} \text{ and } |G| = p^{2r+t+5}.$$

Since  $x^p \in Z(G)$ , we may assume  $x^p = a^{ip^2} b^{jp^2}$ . Since

$$|\langle xb^{-jp}, a \rangle| = p^{r+3} < p^{2r+t+3} = \frac{|G|}{p^2},$$

we have  $[xb^{-jp}, a] = 1$  and hence  $xb^{-jp} \in Z(G)$ . By replacing  $x$  with  $xb^{-jp}$  we get  $x^p = a^{ip^2}$ .

If  $[xa^{-ip}, b] = 1$ , then, replacing  $x$  with  $xa^{-ip}$ , we get  $x^p = 1$  and  $G$  is a group of Type (N6).

If  $[xa^{-ip}, b] \neq 1$ , then  $(i, p) = 1$ . Without loss of generality, we may assume  $x^p = a^{p^2}$ . Since  $G \in \mathcal{A}_3$ , we have

$$|\langle xa^{-ip}, b \rangle| = p^{r+t+4} = \frac{|G|}{p^2} = p^{2r+t+3}.$$

It follows that  $r = 1$ ,  $p \geq 3$  and  $G$  is a group of Type (N7).

If  $s = 1$  and  $r \geq 2$ , then

$$Z(M) = \langle a^{p^2} b^{-p^{t+2}} \rangle \times \langle b^{p^2} \rangle, \quad |M| = p^{2r+t+3} \text{ and } |G| = p^{2r+t+4}.$$

Since  $x^p \in Z(G)$ , we may assume  $x^p = (a^{p^2} b^{-p^{t+2}})^i b^{jp^2}$ . Since

$$|\langle xa^{-ip}, b \rangle| = p^{r+t+3} < p^{2r+t+2} = \frac{|G|}{p^2},$$

we have  $[xa^{-ip}, b] = 1$  and hence  $xa^{-ip} \in Z(G)$ . By replacing  $x$  with  $xa^{-ip}$  we get  $x^p = b^{j'p^2}$ . Since

$$|\langle xb^{-j'p}, a \rangle| = p^{r+3} < p^{2r+t+2} = \frac{|G|}{p^2},$$

we have  $[xb^{-j'p}, a] = 1$  and hence  $xb^{-j'p} \in Z(G)$ . By replacing  $x$  with  $xb^{-j'p}$  we get  $x^p = 1$  and  $G$  is a group of Type (N6).

If  $s = 1$  and  $r = 1$ , then

$$p \geq 3 \text{ and } Z(M) = \langle b^{p^2} \rangle, |M| = p^{t+5} \text{ and } |G| = p^{t+6}.$$

Since  $x^p \in Z(G)$ , we may assume  $x^p = b^{jp^2}$ .

If  $[xb^{-jp}, a] = 1$ , then, replacing  $x$  with  $xb^{-jp}$ , we get  $x^p = 1$  and  $G$  is a group of Type (N6).

If  $[xb^{-jp}, a] \neq 1$ , then  $(j, p) = 1$ . Without loss of generality, we may assume  $x^p = b^{p^2}$ . Since  $G \in \mathcal{A}_3$ , we have

$$|\langle xb^{-p}, a \rangle| = p^4 = \frac{|G|}{p^2} = p^{t+4}.$$

It follows that  $t = 0$  and  $G$  is a group of Type (N8).

If  $s = 0$  and  $r \geq 3$ , then

$$Z(M) = \langle a^{p^2} b^{-p^{t+2}} \rangle \times \langle b^{p^2} \rangle, |M| = p^{2r+t+2} \text{ and } |G| = p^{2r+t+3}.$$

Since  $x^p \in Z(G)$ , we may assume  $x^p = (a^{p^2} b^{-p^{t+2}})^i b^{jp^2}$ . Since

$$|\langle xa^{-ip}, b \rangle| = p^{r+t+3} < p^{2r+t+1} = \frac{|G|}{p^2},$$

we have  $[xa^{-ip}, b] = 1$  and hence  $xa^{-ip} \in Z(G)$ . Replacing  $x$  with  $xa^{-ip}$ , we get  $x^p = b^{j'p^2}$ . Since

$$|\langle xb^{-j'p}, a \rangle| = p^{r+3} < p^{2r+t+1} = \frac{|G|}{p^2},$$

we have  $[xb^{-j'p}, a] = 1$  and hence  $xb^{-j'p} \in Z(G)$ . Replacing  $x$  with  $xb^{-j'p}$ , we get  $x^p = 1$  and  $G$  is a group of Type (N6).

If  $s = 1$  and  $r \leq 2$ , then

$$p \geq 3, r = 2, Z(M) = \langle b^{p^2} \rangle, |M| = p^{t+6} \text{ and } |G| = p^{t+7}.$$

Since  $x^p \in Z(G)$ , we may assume  $x^p = b^{jp^2}$ . Since

$$|\langle xb^{-jp}, ab^{-p^t} \rangle| \leq p^4 < p^{t+5} = \frac{|G|}{p^2},$$

we have  $[xb^{-jp}, a] = 1$  and hence  $xb^{-jp} \in Z(G)$ . By replacing  $x$  with  $xb^{-jp}$  we get  $x^p = 1$  and  $G$  is a group of Type (N6).

Subcase 1.2:  $M$  is the group of Type (18) in Lemma 2.5. That is,  $M = \langle a, b, c \mid a^{p^2} = b^{p^2} = c^p = 1, [a, b] = c, [c, a] = b^{\nu p}, [c, b] = a^p \rangle$ , where  $p \geq 5$ ,  $\nu$  is a fixed quadratic non-residue modulo  $p$ .

In this subcase,  $Z(M) = \langle a^p, b^p \rangle$ ,  $|M| = p^5$  and  $|G| = p^6$ . Since  $x^p \in Z(G)$ , we may assume  $x^p = a^{ip} b^{jp}$ . Since  $|\langle c, xa^{-i} b^{-j} \rangle| \leq p^3$ , we have  $[c, xa^{-i} b^{-j}] = 1$ . It follows that  $x^p = 1$ . Hence we get the group (N9).

Subcase 1.3:  $M$  is one of the groups (19)–(21) in Lemma 2.5.

By an argument similar to that of subcase 1.2, we get groups (N10)–(N12) respectively. The details is omitted.

**Case 2:**  $Z(G) < \Phi(G)$  and  $G$  has at least two three-generator maximal subgroups.

Let  $M_1, \dots, M_s$  be all three-generator maximal subgroups of  $G$ ,  $N = M'_1 M'_2 \dots M'_s$  and  $\bar{G} = G/N$ . Then  $\bar{G}$  has at least two abelian subgroups of index  $p$  and every non-abelian subgroup of  $\bar{G}$  is generated by two elements. Since  $\Phi(G) \not\leq Z(G)$ ,  $\bar{G}$  is not abelian. By Lemma 2.13 we have  $\bar{G} \in \mathcal{A}_2$ . Hence  $\bar{G}$  is a group of Type (8)–(12) in Lemma 2.5. Let  $\bar{G} = \langle \bar{a}, \bar{b}, \bar{x} \rangle$  and  $K = \langle a, b, \Phi(G) \rangle$ . Then  $d(K) = 2$  since  $\bar{K}$  is not abelian. Since  $d(G) = 3$ , we have  $\Phi(K) = \Phi(G)$ . It follows that  $K = \langle a, b \rangle$  is maximal in  $G$  and  $K \in \mathcal{A}_2$ . Hence  $K$  is a group of Type (1)–(7) or (17)–(21) in Lemma 2.5.

Subcase 2.1:  $K$  is a group of Type (1) in Lemma 2.5. That is,  $K = \langle a, b \mid a^8 = b^{2^m} = 1, a^b = a^{-1} \rangle$ .

If  $m = 1$ , then  $\Phi(G) = \Phi(K) = \langle a^2 \rangle$  and  $N = \langle a^4 \rangle$ . By calculation, a maximal subgroup  $\langle a, c \rangle$  of  $G$  is abelian or minimal non-abelian, a contradiction. Hence

$$m \geq 2, \Phi(G) = \Phi(K) = \langle a^2, b^2 \rangle, \langle a^4 \rangle \leq N \leq \langle a^4, b^{2^{m-1}} \rangle \text{ and } |G| = 2^{m+4}.$$

Let  $M = \langle x, a, b^2 \rangle$ . Then  $M \in \mathcal{A}_2$ . Since  $x^2 \in Z(K) \cap \Phi(M)$ , we may assume that  $x^2 = a^{4i} b^{4j}$ . By replacing  $x$  with  $xa^{2i}b^{-2j}$  we get  $x^2 = 1$ . Since  $[x, b^2] = [a, b^2] = 1$ ,  $[x, a] \neq 1$ . Since  $G \in \mathcal{A}_3$ ,  $|\langle x, a \rangle| = 2^{m+2}$ . Since  $|\langle x, a \rangle| \leq 2^5$ ,  $m \leq 3$ .

If  $m = 2$ , then  $N = \langle a^4 \rangle$ . We have  $[x, a] = a^4$  and  $[x, b] = a^{4k}$ . By replacing  $b$  with  $ba^k$  we get  $[x, b] = 1$ . Hence  $G$  is the group (N13).

If  $m = 3$ , then  $|\langle x, a \rangle| = 2^5$ . It follows that  $N = \langle a^4, b^4 \rangle$  and we may assume that  $[x, a] = b^4 a^{4i}$ . If  $[x, a] = b^4 a^4$ , then  $[x, b] = b^4$  or  $[x, ba] = b^4$ . It follows that  $|\langle x, b \rangle| = 2^4$  or  $|\langle x, ba \rangle| = 2^4$ , a contradiction. Hence  $[x, a] = b^4$ . By suitable replacement we get  $[x, b] = a^4$ . Hence  $G$  is the group (N14).

Subcase 2.2:  $K$  is a group of Type (2) in Lemma 2.5. That is,  $K = \langle a, b \mid a^8 = b^{2^m} = 1, a^b = a^3 \rangle$ .

If  $m = 1$ , then  $\Phi(G) = \Phi(K) = \langle a^2 \rangle$  and  $N = \langle a^4 \rangle$ . By calculation, a maximal subgroup  $\langle a, c \rangle$  of  $G$  is abelian or minimal non-abelian, a contradiction. Hence

$$m \geq 2, \Phi(G) = \Phi(K) = \langle a^2, b^2 \rangle, \langle a^4 \rangle \leq N \leq \langle a^4, b^{2^{m-1}} \rangle \text{ and } |G| = 2^{m+4}.$$

Let  $M = \langle x, a, b^2 \rangle$ . Then  $M \in \mathcal{A}_2$ . Since  $x^2 \in Z(K) \cap \Phi(M)$ , we may assume  $x^2 = a^{4i} b^{4j}$ . By replacing  $x$  with  $xa^{2i}b^{-2j}$  we get  $x^2 = 1$ . Since  $[x, b^2] = [a, b^2] = 1$ ,  $[x, a] \neq 1$ . If  $[x, a] = a^4$ , then, replacing  $b$  with  $bx$ , it is reduced to Subcase 2.1. Hence  $|\langle x, a \rangle| = 2^5$ ,  $m = 3$ ,  $N = \langle a^4, b^4 \rangle$  and we may assume  $[x, a] = b^4 a^{4i}$ . If  $[x, a] = b^4 a^4$ , then  $[x, b] = b^4$  or  $[x, ba] = b^4$ . It follows that  $|\langle x, b \rangle| = 2^4$  or  $|\langle x, ba \rangle| = 2^4$ ,

a contradiction. Hence  $[x, a] = b^4$ . By suitable replacement we get  $[x, b] = a^4$ . By replacing  $a$  and  $b$  with  $ax$  and  $bx$ , respectively, we get the group (N14) again.

Subcase 2.3:  $K$  is a group of Type (3) in Lemma 2.5. That is,  $K = \langle a, b \mid a^8 = 1, b^{2^m} = a^4, a^b = a^{-1} \rangle$ .

In this subcase,  $\Phi(G) = \Phi(K) = \langle a^2, b^2 \rangle$  and  $N = \langle a^4 \rangle$ . Let  $M = \langle x, a, b^2 \rangle$ . Then  $M \in \mathcal{A}_2$ . Since  $x^2 \in Z(K) \cap \Phi(M)$ , we may assume that  $x^2 = b^{4i}$ . By replacing  $x$  with  $xb^{-2j}$  we get  $x^2 = 1$ . Since  $[x, b^2] = [a, b^2] = 1$ ,  $[x, a] = a^4$ . It follows that  $m = 2$ . By suitable replacement we get  $[x, b] = 1$ . Hence  $G$  is the group (N15).

Subcase 2.4:  $K$  is a group of Type (4) in Lemma 2.5. That is,  $K = \langle a_1, b; a_2, a_3 \mid a_1^p = a_2^p = a_3^p = b^{p^m} = 1, [a_1, b] = a_2, [a_2, b] = a_3, [a_3, b] = 1, [a_i, a_j] = 1 \rangle$ , where  $p \geq 5$  for  $m = 1$ ,  $p \geq 3$  and  $1 \leq i, j \leq 3$ .

If  $m = 1$ , then  $p \geq 5$  and  $N = \langle a_3 \rangle$ . Let  $M = \langle x, a_1, a_2 \rangle$ . Then  $M \in \mathcal{A}_2$ . Since  $[x, a_2] = [a_1, a_2] = 1$ ,  $[x, a_1] \neq 1$ . Without loss of generality assume  $[x, a_1] = a_3$ . By suitable replacement we get  $[x, b] = 1$ .

If  $x^p = 1$ , then  $G$  is the group (N16).

If  $x^p \neq 1$ , then we may assume that  $a_3 = x^{ip}$ , where  $(i, p) = 1$ . By replacing  $a_1$  and  $a_2$  with  $a_1^{i^{-1}}$  and  $[a_1^{i^{-1}}, b]$ , respectively, we get  $[a_2, b] = [x, a_1] = x^p$ . Hence  $G$  is the group (N17).

If  $m \geq 2$ , then

$$\Phi(G) = \Phi(K) = \langle a_2, a_3, b^p \rangle \text{ and } \langle a_3 \rangle \leq N \leq \langle a_3, b^{p^{m-1}} \rangle.$$

Let  $M = \langle x, a_1, a_2, a_3, b^p \rangle$ . Then  $M \in \mathcal{A}_2$ . Since  $x^p \in Z(K) \cap \Phi(M)$ , we may assume  $x^p = a_3^i b^{jp}$ . If  $(j, p) = 1$ , then  $\langle a_1, xb^{-j} \rangle \in \mathcal{A}_2$  and is of order  $p^4$ , a contradiction. Hence  $p \mid j$ . By replacing  $x$  with  $xb^{-j}$  we get  $x^p = a_3^i$ . Since  $[x, a_2] = [a_1, a_2] = 1$ ,  $[x, a_1] \neq 1$ . Since  $|[x, a_1]| \leq p^4$ , we have  $m = 2$ ,  $(i, p) = 1$  and  $[x, a_1] \notin \langle a_3 \rangle$ . It follows that  $N = \langle a_3, b^p \rangle$ . Without loss of generality assume that  $[a_2, b] = x^p$  and  $[x, a_1] = b^{jp} x^{kp}$ , where  $(j, p) = 1$ . Let  $[x, b] = b^{sp} x^{tp}$ . By replacing  $x$  and  $b$  with  $xa_2^{j^{-1}s-t}$  and  $ba_1^{-j^{-1}s}$ , respectively, we get  $[x, b] = 1$ . By replacing  $b$  with  $bx^{j^{-1}k}$  we get  $[x, a_1] = b^{jp}$ . By replacing  $b$  and  $x$  with  $b^{j^{-1}}$  and  $x^{j^{-2}}$ , respectively, we get  $[x, a_1] = b^p$ . Hence  $G$  is the group (N18).

Subcase 2.5:  $K$  is a group of Type (5) in Lemma 2.5. That is,  $K = \langle a_1, b; a_2 \mid a_1^p = a_2^p = b^{p^{m+1}} = 1, [a_1, b] = a_2, [a_2, b] = b^{p^m}, [a_1, a_2] = 1 \rangle$ , where  $p > 2$ .

In this subcase,  $N = \langle b^{p^m} \rangle$ . Let  $M = \langle x, a_1, a_2, b^p \rangle$ . Then  $M \in \mathcal{A}_2$ . Since  $[x, a_2] = [a_1, a_2] = 1$ ,  $[x, a_1] \neq 1$ . Without loss of generality assume  $[x, a_1] = b^{p^m}$ . Since  $x^p \in Z(K)$ , we may assume  $x^p = b^{ip}$ . If  $(i, p) = 1$ , then  $\langle a_1, xb^{-i} \rangle \in \mathcal{A}_2$  and is of order  $p^4$ . If  $p \mid i$ , then  $\langle a_1, xb^{-i} \rangle \in \mathcal{A}_1$  and is of order  $p^3$ . Hence  $m = 1$  and  $|G| = p^5$ . By suitable replacement we get  $[x, b] = 1$ .

If  $x^p = 1$ , then  $G$  is the group of Type (N19).



If  $x^p = b^{ip} \neq 1$ , then, replacing  $a_1$  and  $b$  with  $a_1^i$  and  $bx^{-i-1}$ , respectively, we get  $b^p = 1$  and  $[a_2, b] = [x, a_1] = x^p$ . Hence  $G$  is the group (N17).

Subcase 2.6:  $K$  is a group of Type (6) in Lemma 2.5. That is,  $K = \langle a_1, b \mid a_1^{p^2} = a_2^p = b^{p^m} = 1, [a_1, b] = a_2, [a_2, b] = a_1^{\nu p}, [a_1, a_2] = 1 \rangle$ , where  $p > 2$  and  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ .

If  $m = 1$ , then  $N = \langle a_1^p \rangle$ . Let  $M = \langle x, a_1, a_2 \rangle$ . Then  $M \in \mathcal{A}_2$ . Since  $[x, a_2] = [a_1, a_2] = 1$ ,  $[x, a_1] \neq 1$ . Without loss of generality assume  $[x, a_1] = a_1^p$ . By suitable replacement we get  $[x, b] = 1$ .

If  $x^p = 1$ , then  $G$  is the group (N20).

If  $x^p = a_1^{ip} \neq 1$ , then, by suitable replacement, we get the group (N17).

If  $m \geq 2$ , then

$$\Phi(G) = \Phi(K) = \langle a_2, a_1^p, b^p \rangle \text{ and } \langle a_1^p \rangle \leq N \leq \langle a_3, b^{p^{m-1}} \rangle.$$

Let  $M = \langle x, a_1, a_2, b^p \rangle$ . Then  $M \in \mathcal{A}_2$ . Since  $x^p \in Z(K) \cap \Phi(M)$ , we may assume that  $x^p = a_1^{ip} b^{jp}$ . If  $(j, p) = 1$ , then  $\langle a_1, x b^{-j} \rangle \in \mathcal{A}_2$  and is of order  $p^4$ , a contradiction. Hence  $p \mid j$ . By replacing  $x$  with  $x b^{-j}$  we get  $x^p = a_1^{ip}$ . If  $x^p \neq 1$ , then, replacing  $a_1$  with  $a_1 x^{-i-1}$ , it is reduced to Subcase 2.4. Hence we may assume  $x^p = 1$ . Since  $[x, a_2] = [a_1, a_2] = 1$ , we have  $[x, a_1] \neq 1$ . Since  $|\langle x, a_1 \rangle| \leq p^4$ , we have  $m = 2$  and  $[x, a_1] \notin \langle a_1^p \rangle$ . It follows that  $N = \langle a_1^p, b^p \rangle$  and  $[x, a_1] = b^{jp} a_1^{kp}$ , where  $(j, p) = 1$ . Let  $[x, b] = b^{sp} a_1^{tp}$ . By replacing  $x$  with  $x a_2^{-\nu^{-1}t}$  we get  $[x, b] = b^{sp}$ . Since  $|\langle x, b \rangle| = p^3$ ,  $[x, b] = 1$ . Assume  $(b a_1)^p = b^p a_1^{rp}$ . By calculation we have  $[x^{j-1} a_2^{r-j^{-1}k}, b a_1] = (b a_1)^p$ . It follows that  $\langle x^{j-1} a_2^{r-j^{-1}k}, b a_1 \rangle \in \mathcal{A}_1$  and is of order  $p^3$ , a contradiction.

Subcase 2.7:  $K$  is a group of Type (7) in Lemma 2.5. That is,  $K = \langle a_1, b; a_2 \mid a_1^9 = a_2^3 = 1, b^3 = a_1^3, [a_1, b] = a_2, [a_2, b] = a_1^{-3} \rangle$ .

In this subcase,  $N = \langle a^3 \rangle = \langle b^3 \rangle$ . If  $x^p = b^{3i} \neq 1$ , then, replacing  $b$  with  $bx^{-i}$ , it is reduced to Subcase 2.6. Hence we may assume  $x^3 = 1$ . Let  $M = \langle x, a_1, a_2 \rangle$ . Then  $M \in \mathcal{A}_2$ . Since  $[x, a_2] = [a_1, a_2] = 1$ ,  $[x, a_1] \neq 1$ . Without loss of generality assume  $[x, a_1] = a_1^3$ . By suitable replacement we get  $[x, b] = 1$ . Hence  $G$  is the group (N21).

Subcase 2.8:  $K$  is a group of Type (17) in Lemma 2.5. That is,  $K = \langle a, b \mid a^{p^{r+2}} = 1, b^{p^{r+s+t}} = a^{p^{r+s}}, [a, b] = a^{p^r} \rangle$ , where  $r \geq 2$  for  $p = 2$ ,  $r \geq 1$  for  $p \geq 3$ ,  $t \geq 0$ ,  $0 \leq s \leq 2$  and  $r + s \geq 2$ .

In this subcase,

$$Z(K) = \langle a^{p^2}, b^{p^2} \rangle, |K| = p^{2r+s+t+2} \text{ and } |G| = p^{2r+s+t+3}.$$

If  $N = \langle a^{p^{r+1}} \rangle$ , then, assuming that  $[x, a] = a^{ip^{r+1}}$  and  $[x, b] = a^{jp^{r+1}}$ , we get  $x a^{-jp} b^{ip} \in Z(G)$ , a contradiction. Hence  $|N| = p^2$ . Since  $x^p \in Z(G)$ , we may assume that  $x^p = a^{ip^2} b^{jp^2}$ . By replacing  $x$  with  $x a^{-ip} b^{-jp}$  we get  $x^p = 1$ .

If  $s = 2$ , then

$$Z(K) = \langle a^{p^2} \rangle \times \langle b^{p^2} \rangle, \quad |K| = p^{2r+t+4}, \quad |G| = p^{2r+t+5} \text{ and } N = \langle a^{p^{r+1}}, b^{p^{r+t+1}} \rangle.$$

Without loss of generality assume  $[x, b] \neq 1$ . Since  $|\langle x, b \rangle| \leq p^{r+t+4}$ ,  $r+t+4 \geq 2r+t+3$ . It follows that  $r = 1$ ,  $p \geq 3$  and  $[x, b] \notin \langle b^{p^{t+2}} \rangle$ . We may assume  $[x, b] = a^{p^2} b^{kp^{t+2}}$ .

If  $[x, a] \neq 1$ , then  $t = 0$ ,  $|G| = p^7$  and  $[x, a] \notin \langle a^{p^2} \rangle$ . By suitable replacement we may assume that  $[x, b] = a^{p^2}$  and  $[x, a] = b^\nu p^2 a^{kp^2}$ , where  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ . By Lemma 5.14 we have  $G$  is the group (N22).

If  $[x, a] = 1$ , then  $(k, p) = 1$  since  $xa^{-p} \notin Z(G)$ . By replacing  $a$  and  $x$  with  $a^{k^{-1}}$  and  $x^{k^{-1}}$ , respectively, we have  $[x, b] = a^{p^2} b^{p^{t+2}}$ . If  $t = 0$ , then  $\langle x, ba \rangle$  is not abelian and is of order  $p^4$ , a contradiction. Hence  $t \geq 1$  and  $G$  is the group (N23).

If  $s = 1$ , then

$$|K| = p^{2r+t+3}, \quad |G| = p^{2r+t+4}, \quad r \geq 2 \text{ and } N = \langle a^{p^{r+1}}, a^{-p^r} b^{p^{r+t}} \rangle.$$

Since  $|\langle x, a^{-1} b^{p^t} \rangle| \leq p^{r+3} < p^{2r+t+2}$ ,  $[x, a^{-1} b^{p^t}] = 1$ . Since  $Z(G) < \Phi(G)$ ,  $[x, b] \neq 1$ . Since  $|\langle x, b \rangle| = p^{r+t+4} = p^{2r+t+2} = \frac{|G|}{p^2}$ ,  $r = 2$ . Without loss of generality assume  $[x, b] = a^{-p^2} b^{p^{t+2}} b^{ip^{t+3}}$ . By replacing  $a$  with  $axa^{-ip}$  we have  $[a, b] = b^{p^{t+2}}$ . By replacing  $a$  and  $b$  with  $b$  and  $a^{-1} b^{p^t} b^{ip^{t+1}}$ , respectively, we get

$$G = \langle a, b, x \mid a^{p^{t+4}} = b^{p^3} = x^p = 1, [a, b] = a^{p^{t+2}}, [x, b] = 1, [x, a] = b^{p^2} \rangle.$$

Hence  $G$  is a group of Type (N24).

If  $s = 0$ , then  $r \geq 2$ . By suitable replacement we have

$$K = \langle a, b \mid a^{p^{r+t+2}} = b^{p^r} = 1, [a, b] = a^{p^{r+t}} \rangle.$$

By calculation we have

$$Z(K) = \langle a^{p^2} \rangle \times \langle b^{p^2} \rangle, \quad |K| = p^{2r+t+2}, \quad |G| = p^{2r+t+3}, \quad r \geq 3 \text{ and } N = \langle a^{p^{r+t+1}}, b^{p^{r-1}} \rangle.$$

Since  $|\langle x, a \rangle| = p^{r+t+4} = \frac{|G|}{p^2}$ ,  $r = 3$ . Without loss of generality assume  $[x, a] = b^{p^2} a^{kp^{t+4}}$ . By replacing  $b$  with  $ba^{kp^{t+2}}$  we get  $[x, a] = b^{p^2}$ . Hence  $G$  is a group of Type (N24).

Subcase 2.9:  $K$  is a group of Type (18) in Lemma 2.5. That is,  $K = \langle a, b \mid a^{p^2} = b^{p^2} = c^p = 1, [a, b] = c, [c, a] = b^\nu p, [c, b] = a^p \rangle$ , where  $p \geq 5$ ,  $\nu$  is a fixed quadratic non-residue modulo  $p$ .

In this subcase,  $|G| = p^6$  and  $N = Z(K) = \langle a^p, b^p \rangle$ . We claim that  $x^p = 1$ . Otherwise, since  $x^p \in Z(K)$ , we may assume that  $x^p = a^{ip} b^{jp}$ , where  $(i, j) \neq (0, 0)$ . If  $i \neq 0$ , then  $\langle xa^{-i}, c \rangle$  is not abelian and is of order  $p^3$ , a contradiction. If  $j \neq 0$ , then  $\langle xb^{-j}, c \rangle$  is not abelian and is of order  $p^3$ , a contradiction again. Hence  $x^p = 1$ . Let  $[x, a] = a^{sp} b^{tp}$ . Since  $|\langle xc^{-\nu^{-1}t}, a \rangle| = p^3$ , we have  $[xc^{-\nu^{-1}t}, a] = 1$  and hence  $[x, a] = b^{tp}$ .

By replacing  $x$  with  $xc^{-\nu^{-1}t}$  we get  $[x, a] = 1$ . The same reason gives that  $[x, b] = a^{up}$ . Since  $Z(G) < \Phi(G)$ ,  $(u, p) = 1$ . Without loss of generality assume  $[x, b] = a^p$ . Then  $\langle cx^{\nu-1}, ab \rangle$  is not abelian and is of order  $p^3$ , a contradiction.

Subcase 2.10:  $K$  is a group of Type (19)–(21) in Lemma 2.5. That is,  $K = \langle a, b; c \mid a^{p^2} = b^{p^2} = c^p = 1, [a, b] = c, [c, a] = b^{\nu p}, [c, b] = a^p \rangle$ , where  $p \geq 5$ ,  $\nu$  is a fixed quadratic non-residue modulo  $p$ .

By an argument similar to that of subcase 2.9, we get a contradiction, respectively. The details is omitted.

**Case 3:**  $Z(G) < \Phi(G)$  and  $M$  is the unique three-generator maximal subgroup of  $G$ .

Then  $G/M'$  has a unique abelian subgroup  $M/M'$  of index  $p$ , and all non-abelian subgroups of  $G/M'$  are generated by two elements. By Lemma 2.13,  $G/M' \in \mathcal{A}_2$ . Hence  $G$  a group of Type (13)–(16) in Lemma 2.5.

Subcase 3.1:  $G/M'$  is a group of Type (13) in Lemma 2.5. That is,  $G/M' = \langle \bar{a}, \bar{b}, \bar{c} \mid \bar{a}^4 = \bar{b}^4 = 1, \bar{c}^2 = \bar{a}^2\bar{b}^2, [\bar{a}, \bar{b}] = \bar{b}^2, [\bar{c}, \bar{a}] = \bar{a}^2, [\bar{c}, \bar{b}] = 1 \rangle$ .

Let  $K = \langle a, b, \Phi(G) \rangle$ . Since  $K/M'$  is not abelian, we have  $d(K) = 2$  and  $K \in \mathcal{A}_2$ . Hence  $\Phi(K) = \Phi(G)$  and  $K = \langle a, b \rangle$ . By calculation we get

$$[c, a^2] = [c, a]^2[c, a, a] = a^4 \text{ and } [c^2, a] = [c, a]^2[c, a, c] = a^8 = 1.$$

Since  $c^2 \equiv a^2b^2 \pmod{M'}$ ,  $[a, b^2] = 1$ . It follows that  $b^4 = [a, b^2] = 1$  and hence  $\langle a, b \rangle \in \mathcal{A}_1$ , a contradiction.

Subcase 3.2:  $G/M'$  is a group of Type (14) in Lemma 2.5. That is,  $G/M' = \langle \bar{a}, \bar{b}, \bar{d} \mid \bar{a}^{p^m} = \bar{b}^{p^2} = \bar{d}^p = 1, [\bar{a}, \bar{b}] = \bar{a}^{p^{m-1}}, [\bar{d}, \bar{a}] = \bar{b}^p, [\bar{d}, \bar{b}] = 1 \rangle$ , where  $m \geq 3$  if  $p = 2$ .

Let  $K = \langle a, b, \Phi(G) \rangle$  and  $L = \langle d, a, \Phi(G) \rangle$ . Since  $K/M'$  and  $L/M'$  is not abelian, we have

$$d(K) = 2, d(L) = 2, K \in \mathcal{A}_2 \text{ and } L \in \mathcal{A}_2.$$

Hence

$$\Phi(K) = \Phi(L) = \Phi(G), K = \langle a, b \rangle \text{ and } L = \langle d, a \rangle.$$

By calculation,  $b^{p^2} = [d^p, a] = 1$ . Since  $L \in \mathcal{A}_2$ ,  $[a, b^p] \neq 1$ . It follows that  $a^{p^m} = [a, b^p] \neq 1$  and hence  $o(a) = p^{m+1}$ . Let  $N = \langle a^p, b \rangle$ . Then  $N \in \mathcal{A}_1$ . Since  $|N| = p^{m+2}$ , we get  $|G| = p^{m+4}$  and hence  $M' = \langle a^{p^m} \rangle$ . We claim that  $p > 2$ . Otherwise,  $\langle d, a^2 \rangle$  is not abelian and is of order  $2^{m+1}$ , a contradiction. Hence  $p > 2$ . Let  $d^p = a^{ip^m}$ . By replacing  $d$  with  $da^{-ip^{m-1}}$  we get  $d^p = 1$ . By suitable replacement we get  $[a, b] = a^{p^{m-1}}$  and  $[d, a] = b^p$ .

If  $[d, b] = 1$  then  $G$  is the group (N25).

If  $[d, b] \neq 1$ , then, since  $|\langle d, b \rangle| = p^4$ , we have  $m = 2$ . Assume that  $[d, b] = a^{jp^2}$ , where  $j = k^2\nu$ ,  $\nu = 1$  or a fixed quadratic non-residue modulo  $p$ . By replacing  $a$  and  $d$  with  $a^k$  and  $d^{k^{-1}}$ , respectively, we get  $[d, b] = a^{\nu p^2}$ . Hence  $G$  is the group (N26).

Subcase 3.3:  $G/M'$  is a group of Type (15) in Lemma 2.5. That is,  $\langle \bar{a}, \bar{b}, \bar{d} \mid \bar{a}^{p^m} = \bar{b}^{p^2} = \bar{d}^{p^2} = 1, [\bar{a}, \bar{b}] = \bar{d}^p, [\bar{d}, \bar{a}] = \bar{b}^{jp}, [\bar{d}, \bar{b}] = 1 \rangle$ , where  $(j, p) = 1$ ,  $p > 2$ ,  $j$  is a fixed quadratic non-residue modulo  $p$ , and  $-4j$  is a quadratic non-residue modulo  $p$ .

Let  $K = \langle a, b, \Phi(G) \rangle$  and  $L = \langle d, a, \Phi(G) \rangle$ . Since  $K/M'$  and  $L/M'$  is not abelian, we have

$$d(K) = 2, \quad d(L) = 2, \quad K \in \mathcal{A}_2 \text{ and } L \in \mathcal{A}_2.$$

Hence

$$\Phi(K) = \Phi(L) = \Phi(G), \quad K = \langle a, b \rangle \text{ and } L = \langle d, a \rangle.$$

By Lemma 2.6 (7),  $\exp(K') = \exp(L') = p$ . It follows that  $d^{p^2} = b^{p^2} = 1$ . By calculation we have  $[d^p, b] = [d, b]^p = 1$  and  $[d^p, a] = [d, a]^p = 1$ . Hence  $K \in \mathcal{A}_1$ , a contradiction.

Subcase 3.4:  $G/M'$  is a group of Type (16) in Lemma 2.5. That is,  $\langle \bar{a}, \bar{b}, \bar{d} \mid \bar{a}^{p^m} = \bar{b}^{p^2} = \bar{d}^{p^2} = 1, [\bar{a}, \bar{b}] = \bar{d}^p, [\bar{d}, \bar{a}] = \bar{b}^{jp}\bar{d}^p, [\bar{d}, \bar{b}] = 1 \rangle$ , where if  $p$  is odd, then  $4j = 1 - \rho^{2r+1}$  with  $1 \leq r \leq \frac{p-1}{2}$  and  $\rho$  the smallest positive integer which is a primitive root (mod  $p$ ); if  $p = 2$ , then  $j = 1$ .

Let  $K = \langle a, b, \Phi(G) \rangle$  and  $L = \langle d, a, \Phi(G) \rangle$ . Since  $K/M'$  and  $L/M'$  is not abelian, we have

$$d(K) = 2, \quad d(L) = 2, \quad K \in \mathcal{A}_2 \text{ and } L \in \mathcal{A}_2.$$

Hence

$$\Phi(K) = \Phi(L) = \Phi(G), \quad K = \langle a, b \rangle \text{ and } L = \langle d, a \rangle.$$

By Lemma 2.6 (7),  $\exp(K') = \exp(L') = p$ . It follows that  $d^{p^2} = b^{p^2} = 1$ . By calculation we have  $[d^p, b] = [d, b]^p = 1$  and  $[d^p, a] = [d, a]^p = 1$ . Hence  $K \in \mathcal{A}_1$ , a contradiction.

We calculate the  $(\mu_0, \mu_1, \mu_2)$  and  $\alpha_1(G)$  of those groups in Theorem 5.15 as follows. Since  $d(G) = 3$ ,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1 + p + p^2)$ . In the following, we calculate  $\alpha_1(G)$ .

**Case 1.**  $G$  is one of the groups (N1)–(N5).

Since  $\Phi(G) \leq Z(G)$ , by Theorem 3.18,  $\alpha_1(G) = \mu_1 + \mu_2 p^2 = p^4 + p^3 + p^2$ .

**Case 2.**  $G$  is one of the groups (N6)–(N12).

In this case,  $G = M * \langle x \rangle$  where  $M = \langle a, b \rangle$  such that  $\alpha_1(M) = 1 + p$  and  $x^p \in Z(M)$ .

Other maximal subgroups of  $G$  are:

$N_i = M_i \langle x \rangle$  where  $M_i < M$ ;

$N_{ij} = \langle ax^i, bx^j, \Phi(M) \rangle$  where  $0 \leq i, j \leq p - 1$ .

Since  $d(N_i) = 3$  and  $N'_i = M'_i$ , by Lemma 2.6 (7),  $\alpha_1(N_i) = p^2$ . Notice that  $G = N_{ij} * \langle x \rangle$ . If  $d(N_{ij}) = 3$ , then  $G' = N'_{ij} \leq Z(G)$  and  $\exp(G') = p$ . It follows that  $\Phi(G) \leq Z(G)$ , a contradiction. Hence  $d(N_{ij}) = 2$ . If  $N_{ij}$  has an abelian subgroup

of index  $p$ , then  $G$  also has an abelian subgroup of index  $p$ , a contradiction. Hence  $\alpha_1(N_{ij}) = 1 + p$ . Let  $H \in \Gamma_2$ . Then  $H = \langle xm, \Phi(G) \rangle$  where  $m \in M$  or  $H = \langle m, \Phi(G) \rangle$  where  $m \in M \setminus \Phi(G)$ . It is obvious that  $H' = 1$  if and only if  $H = \langle x, \Phi(G) \rangle$ . Thus  $\sum_{H \in \Gamma_2} \alpha_1(H) = p + p^2$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) - p \sum_{H \in \Gamma_2} \alpha_1(H) = (1 + p) \times p^2 + p^2 \times (1 + p) - p \times (p + p^2) = p^3 + p^2.$$

**Case 3.**  $G$  is one of the groups (N13)–(N21) except for (N14) and (N18).

In this case,  $G = M * \langle x \rangle$  where  $\langle x \rangle \cong C_p$  and  $M = \langle a, b \rangle$  such that  $d(M) = 2$ ,  $|M'| = p^2$ ,  $[x, b] = 1$ ,  $[x, a] \in M_3$  and  $\langle a, \Phi(M) \rangle$  is the unique abelian maximal subgroup of  $G$ . Hence all maximal subgroups are:

$N_i = K_i * \langle x \rangle$  where  $K_i$  are maximal subgroups of  $M$ ;

$N_{ij} = \langle ax^i, bx^j \rangle$  where  $0 \leq i, j \leq p - 1$ .

It is easy to see that  $K'_i = M_3$  and hence  $|N'_i| = p$ . By Lemma 2.6,  $\alpha_1(N_i) = p^2$ . Since  $N_{ij} \cong M$ ,  $\alpha_1(N_{ij}) = p$ . Let  $H \in \Gamma_2$ . Then  $H = \langle a^i b^j x^k, \Phi(G) \rangle$ . It is obvious that  $H \in \mathcal{A}_1$  if and only if  $j \neq 0$ . Hence  $\sum_{H \in \Gamma_2} \alpha_1(H) = p^2$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) - p \sum_{H \in \Gamma_2} \alpha_1(H) = (1 + p) \times p^2 + p^2 \times p - p \times p^2 = p^3 + p^2.$$

**Case 4.**  $G$  is either the group (N14) or (N18).

In this case,  $G = M * \langle x \rangle$  where  $\langle x \rangle \cong C_p$  and  $M = \langle a, b \rangle$  such that  $d(M) = 2$ ,  $|M'| = p^2$ ,  $[x, b] \in M_3$ ,  $[x, a] \notin M'$  and  $\langle a, \Phi(M) \rangle$  is the unique abelian maximal subgroup of  $G$ . Hence all maximal subgroups are:

$N = \langle b, x, \Phi(M) \rangle$ ;

$N_i = \langle ab^i, x, \Phi(M) \rangle$  where  $0 \leq i, j \leq p - 1$ .

$N_{ij} = \langle ax^i, bx^j \rangle$  where  $0 \leq i, j \leq p - 1$ .

It is easy to see that  $N' = M_3$ . By Lemma 2.6 (7),  $\alpha_1(N) = p^2$ . By calculation,  $|N'_0| = p$  and  $|N'_i| = p^2$  for  $i \neq 0$ . Hence  $\alpha_1(N_0) = p^2$  and  $\alpha_1(N_i) = p^2 + p$  for  $i \neq 0$ . Since  $N_{ij} \cong M$ ,  $\alpha_1(N_{ij}) = p$ . Let  $H \in \Gamma_2$ . Then  $H = \langle a^i b^j x^k, \Phi(G) \rangle$ . It is obvious that  $H \in \mathcal{A}_1$  if and only if  $j \neq 0$ . Hence  $\sum_{H \in \Gamma_2} \alpha_1(H) = p^2$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) - p \sum_{H \in \Gamma_2} \alpha_1(H) = 2 \times p^2 + (p - 1) \times (p^2 + p) + p^2 \times p - p \times p^2 = p^3 + 2p^2 - p.$$

**Case 5.**  $G$  is one of the groups (N22)–(N24).

In this case,  $p > 2$ ,  $G = M * \langle x \rangle$  where  $\langle x \rangle \cong C_p$  and  $M = \langle a, b \rangle$  such that  $M$  is metacyclic,  $[x, b] \in \Phi(M')$ ,  $[x, a] \notin M'$ . Hence all maximal subgroups are:

$N = \langle b, x, \Phi(M) \rangle$ ;

$N_i = \langle ab^i, x, \Phi(M) \rangle$  where  $0 \leq i, j \leq p - 1$ .

$N_{ij} = \langle ax^i, bx^j \rangle$  where  $0 \leq i, j \leq p-1$ .

It is easy to see that  $N' = \Phi(M')$ . By Lemma 2.6 (7),  $\alpha_1(N) = p^2$ . By calculation,  $|N'_i| = p^2$ . Hence  $\alpha_1(N_i) = p^2 + p$ . Since  $N_{ij} \cong M$ ,  $\alpha_1(N_{ij}) = 1 + p$ . Let  $H \in \Gamma_2$ . Then  $H = \langle a^i b^j x^k, \Phi(G) \rangle$ . It is obvious that  $H \notin \mathcal{A}_1$  if and only if  $i = j = 0$ . Hence  $\sum_{H \in \Gamma_2} \alpha_1(H) = p^2 + p$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) - p \sum_{H \in \Gamma_2} \alpha_1(H) = p^2 + p \times (p^2 + p) + p^2 \times (1 + p) - p \times (p^2 + p) = p^3 + 2p^2.$$

**Case 6.**  $G$  is one of the groups (N25)–(N26).

It is easy to verify that  $M = \langle b, d, a^p \rangle$  is the unique three-generator maximal subgroup of  $G$ . Since  $|M'| = p$ , by Lemma 2.6,  $\alpha_1(N) = p^2$ . Other maximal subgroups are:

$N_i = \langle ab^i, d \rangle$  where  $0 \leq i \leq p-1$ .

$N_{ij} = \langle ad^i, bd^j \rangle$  where  $0 \leq i, j \leq p-1$ .

It is easy to see that  $N_i$  has a unique abelian maximal subgroup  $\langle d, a^p, b^p \rangle$  and  $N_{ij}$  has no abelian maximal subgroup. Hence  $\alpha_1(N_i) = p$  and  $\alpha_1(N_{ij}) = 1 + p$ . Let  $H \in \Gamma_2$ . Then  $H = \langle a^i b^j d^k, \Phi(G) \rangle$  where  $(i, j, k) \neq (0, 0, 0)$ . It is obvious that  $H \notin \mathcal{A}_1$  if and only if  $i = j = 0$ , hence if and only if  $H = \langle d, \Phi(G) \rangle$ . Thus  $\sum_{H \in \Gamma_2} \alpha_1(H) = p^2 + p$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) - p \sum_{H \in \Gamma_2} \alpha_1(H) = p^2 + p \times p + p^2 \times (1 + p) - p \times (p^2 + p) = 2p^2.$$

□

**Theorem 5.16.** *Suppose that  $G$  is an  $\mathcal{A}_3$ -group having no abelian subgroup of index  $p$ . Then  $d(G) = 4$  if and only if  $G$  is isomorphic to one of the following pairwise non-isomorphic groups:*

(Oi)  $G' \cong C_p$  and  $c(G) = 2$ . In this case,  $\Phi(G) = Z(G) = G'$ ,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1 + p + p^2 + p^3)$  and  $\alpha_1(G) = p^2 + p^4$ .

(O1)  $D_8 * Q_8$ ;

(O2)  $Q_8 * Q_8$ ;

(O3)  $M_p(1, 1, 1) * M_p(2, 1)$ , where  $p > 2$ ;

(O4)  $M_p(1, 1, 1) * M_p(1, 1, 1)$ , where  $p > 2$ .

(Oii)  $G' \cong C_p^2$  and  $c(G) = 2$ .

(O5)  $G = \langle a, b, c, d \mid a^4 = b^4 = 1, c^2 = a^2, d^2 = b^2, [a, b] = 1, [a, c] = b^2, [b, c] = a^2, [a, d] = a^2, [b, d] = a^2 b^2, [c, d] = 1 \rangle$ ; here  $|G| = 2^6$ ,  $\Phi(G) = Z(G) = G' = \langle a^2, b^2 \rangle \cong C_2^2$ , any two noncommutative elements of  $G$  generate  $M_2(2, 2)$ .  $(\mu_0, \mu_1, \mu_2) = (0, 0, 15)$  and  $\alpha_1(G) = 30$ .

(Oiii)  $G' \cong C_p^3$  and  $c(G) = 2$ .

(O6)  $G = K \times \langle a_4 \rangle$  where  $K = \langle a_1, a_2, a_3 \mid a_1^4 = a_2^4 = a_3^4 = 1, [a_1, a_2] = a_3^2, [a_1, a_3] = a_2^2 a_3^2, [a_2, a_3] = a_1^2 a_2^2, [a_1^2, a_2] = [a_2^2, a_1] = 1 \rangle$  and  $\langle a_4 \rangle \cong C_2$ ; here  $|G| = 2^7$ ,  $\Phi(G) = G' = \langle a_1^2, a_2^2, a_3^2 \rangle$ ,  $Z(G) = \langle a_1^2, a_2^2, a_3^2, a_4 \rangle \cong C_2^4$ ,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 15)$  and  $\alpha_1(G) = 30$ .

**Proof** Let the type of  $G/G'$  be  $(p^{m_1}, p^{m_2}, p^{m_3}, p^{m_4})$ , where  $m_1 \geq m_2 \geq m_3 \geq m_4$ , and  $G/G' = \langle a_1 G' \rangle \times \langle a_2 G' \rangle \times \langle a_3 G' \rangle \times \langle a_4 G' \rangle$ , where  $o(a_i G') = p^{m_i}$ ,  $i = 1, 2, 3, 4$ . Then  $G = \langle a_1, a_2, a_3, a_4 \rangle$ . By Lemma 3.17,  $c(G) = 2$ ,  $\Phi(G) \leq Z(G)$ ,  $G' \leq C_p^3$  and all  $\mathcal{A}_1$ -subgroups of  $G$  contain  $\Phi(G)$ . The last property gives that  $\sum_{H \in \Gamma_2} \alpha_1(H) = \alpha_1(G)$ .

We claim that  $m_1 = 1$ . Otherwise,  $m_1 \geq 2$ . Let  $B = \langle a_2, a_3, a_4 \rangle$ . Since  $|G : B| \geq p^2$ , we deduce that  $B' = 1$ . Hence  $A = \langle B, a_1^p \rangle$  is an abelian subgroup of index  $p$  of  $G$ , a contradiction. Hence  $G/G'$  is elementary abelian.

**Case 1:**  $G' \cong C_p$ .

In this case,  $|G| = p^5$ . We claim that  $Z(G) = G'$ . Otherwise, without loss of generality assume  $a_1 \in Z(G)$ . Let  $B = \langle a_2, a_3, a_4 \rangle$ . Then  $B' \cong C_p$ . By Lemma 2.6 (3),  $|Z(B)| = p^2$ . Since  $Z(G) \geq \langle a_1, Z(B) \rangle$ ,  $|Z(G)| \geq p^3$ . Hence  $G$  has an abelian subgroup of index  $p$  of  $G$ , a contradiction.

By above argument,  $G$  is an extraspecial  $p$ -group. Hence we get groups (O1)–(O4).

**Case 2:**  $G' \cong C_p^2$ .

In this case,  $|G| = p^6$  and any two noncommutative elements generate an  $\mathcal{A}_1$ -group of order  $p^4$ . Such groups were classified by [4]. By checking those groups listed in [4] we get the group (O5).

**Case 3:**  $G' \cong C_p^3$ .

In this case,  $|G| = p^7$  and any  $\mathcal{A}_1$ -subgroup is  $M_p(2, 2, 1)$ . It follows that  $\Omega_1(G) \leq Z(G)$ . Let  $N \leq G'$  such that  $|N| = p$  and  $\bar{G} = G/N$ . Then  $|\bar{G}'| = p^2$ . By Lemma 2.16, there exists  $\bar{K} \leq \bar{G}$  such that  $d(\bar{K}) = 3$  and  $\bar{K}' = \bar{G}'$ . Without loss of generality assume  $K = \langle a_1, a_2, a_3 \rangle$ . It follows that  $|K'| \geq p^2$  and  $G' = \langle a_1^p, a_2^p, a_3^p \rangle$ .

Subcase 3.1:  $K' \cong C_p^3$ .

By Lemma 2.5,

$$K = \langle a_1, a_2, a_3 \mid a_1^4 = a_2^4 = a_3^4 = 1, [a_1, a_2] = a_3^2, [a_1, a_3] = a_2^2 a_3^2, [a_2, a_3] = a_1^2 a_2^2 \rangle.$$

Since  $a_4^2 \in G' = K'$ , there exists  $c \in K$  such that  $a_4^2 = c^2$ . Since  $|\langle a_4, c \rangle| \leq 16$ ,  $[a_4, c] = 1$ . By replacing  $a_4$  with  $a_4 c$  we get  $a_4^2 = 1$ . Hence  $a_4 \in \Omega_1(G) \leq Z(G)$  and we get the group (O6).

Subcase 3.2:  $K' \cong C_p^2$ .

Without loss of generality assume  $K' = \langle [a_1, a_2], [a_1, a_3] \rangle$  and  $[a_2, a_3] = 1$ . If  $G' = \langle K', [a_1, a_4] \rangle$ , then there exists  $b \in \langle a_2, a_3, a_4 \rangle$  such that  $[a_1, b] = a_1^p$ , which contradicts that  $G' \leq \langle a_1, b \rangle$ . Hence  $[a_1, a_4] \in K'$ . By suitable replacement we get  $[a_1, a_4] = 1$ . Without loss of generality assume  $G' = \langle K', [a_2, a_4] \rangle$ . By replacing  $a_3$  with  $a_3 a_4$  we have  $K' = G'$ . It is reduced to subcase 3.1.

We calculate the  $(\mu_0, \mu_1, \mu_2)$  and  $\alpha_1(G)$  of those groups in Theorem 5.16 as follows.

Since  $d(G) = 4$ ,  $(\mu_0, \mu_1, \mu_2) = (0, 0, 1 + p + p^2 + p^3)$ . In the following, we calculate  $\alpha_1(G)$ .

**Case 1.**  $G$  is one of the groups (O1)–(O4).

Let  $H \in \Gamma_1$ . Then  $|H'| = p$ . By Lemma 2.6 (7),  $\alpha_1(H) = p^2$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) - p \sum_{H \in \Gamma_2} \alpha_1(H) = (1 + p + p^2 + p^3) \times p^2 - p\alpha_1(G).$$

Hence

$$\alpha_1(G) = \frac{(1 + p + p^2 + p^3)p^2}{1 + p} = p^2 + p^4.$$

**Case 2.**  $G$  is the group (O5).

By calculation,  $d(H) = 3$  and  $H' \cong C_2^2$  for any  $H \in \Gamma_1$ . Hence  $\alpha_1(H) = 6$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) - p \sum_{H \in \Gamma_2} \alpha_1(H) = 15 \times 6 - 2\alpha_1(G).$$

Hence  $\alpha_1(G) = 30$ .

**Case 3.**  $G$  is the group (O6).

Let  $H \in \Gamma_1$ . Then  $H$  is one of the following types: (1)  $H = M \times \langle a_4 \rangle$  where  $M < K$ ; (2)  $H = \langle a_1 a_4^i, a_2 a_4^j, a_3 a_4^k \rangle$  where  $i, j, k = 0, 1$ . If  $H$  is of Type (1), then  $|H'| = 2$  and hence  $\alpha_1(H) = 4$  by Lemma 2.6. If  $H$  is of Type (2), then  $H \cong K$  and hence  $\alpha_1(H) = 7$ . By Hall's enumeration principle,

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) - p \sum_{H \in \Gamma_2} \alpha_1(H) = 7 \times 4 + 8 \times 7 - 2\alpha_1(G).$$

Hence  $\alpha_1(G) = 28$ . □

Now we list the triple  $(\mu_0, \mu_1, \mu_2)$  and  $\alpha_1(G)$  for  $\mathcal{A}_3$ -groups in Table 13 and 14 respectively. This solves Problem 893 in [8], Problem 1595 in [9] and Problem 2829 in [10] respectively.

At the end of this paper, we list some enumeration properties of  $\mathcal{A}_3$ -groups, which can easily be obtained from Table 14.



**Theorem 5.17.** *Let  $G$  be an  $\mathcal{A}_3$ -group. Then  $p^2 \leq \alpha_1(G) \leq p^4 + p^3 + p^2 + p$ , and the following conclusions hold.*

- (1) *If  $\alpha_1(G) = p^2$ , then  $d(G) = 2$ ,  $c(G) = 4$  and  $G$  has an abelian maximal subgroup. In this case, non-abelian subgroups of  $G$  are generated by two elements;*
- (2) *If  $\alpha_1(G) = p^4 + p^3 + p^2 + p$ , then  $p = 2$ ,  $c(G) = 2$ ,  $d(G) = 4$  and  $G = K \times C_2$  where  $K$  is the smallest Suzuki 2-group;*
- (3) *If  $p > 2$ , then  $\alpha_1(G) \leq p^4 + p^3 + p^2$ ;*
- (4) *If  $d(G) = 2$ , then  $\alpha_1(G) \leq p^3 + 2p^2 + p$ ;*
- (5) *If  $d(G) = 2$  and  $\alpha_1(G) \leq p^3 + 2p^2 + p$ , then  $p = 2$  and  $G$  is the group (M37) in Theorem 5.13;*
- (6) *If  $d(G) \geq 3$ , then  $\alpha_1(G) \geq 2p^2 - 1$ .*

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$(\mu_0, \mu_1, \mu_2)$	types of $\mathcal{A}_3$ groups
$(1, p-1, 1)$	(A1)–(A6)
$(p+1, p^2-1, 1)$	(B1); (B2) where $m = n = 1$ ; (B3) where $n = 1$ ; (B4); (B5) where $n = 1$
$(p+1, p^2-p, p)$	(B2) where $m > 1 = n$ or $n > 1 = m$ ; (B3) where $n > 1$ ; (B5) where $n > 1$
$(1, p^2-1, p+1)$	(B6); (B9); (B11); (B13) where $p = 2$ and $m = l = 1$ (B17) where $l = 1$ ; (B19) where $p = 2$ and $m = l = 1$
$(1, p^2, p)$	(B7) where $l > 1$ ; (B10); (B12) where $l > 1$ ; (B13) where $m = 1$ and $l > 1$ (B14) where $n = 1$ or $m = 1$ ; (B16); (B18) where $m = 1$ (B19) where $p = 2$ and $l > 1 = m$ ; (B19) where $p > 2$ and $m = 1$
$(1, p^2+p-1, 1)$	(B7) where $l = 1$ ; (B12) where $l = 1$ ; (B13) where $p > 2$ and $m = l = 1$ ; (B20)
$(1, p^2+p-2, 2)$	(B8) where $l = 1$
$(1, p^2-p, 2p)$	(B8) where $l > 1$ ; (B13) where $m = 2$ ; (B14) where $n = m = 2$ (B15); (B17) where $l > 1$ ; (B18) where $m = 2$ ; (B19) where $m = 2$
$(0, p, 1)$	(C1)–(C6), (C8), (C9), (C11), (C15)–(C17)
$(0, p-1, 2)$	(C7), (C10), (C12)–(C14)
$(0, p^2, p+1)$	(D1), (D4), (D6)–(D10), (D12)–(D14), (D16)
$(0, p^2+p, 1)$	(D2) where $-\nu \notin (F_p^*)^2$ ; (D3) where $-r \notin (F_p^*)^2$ ; (D5); (D11) where $-\nu \notin (F_p^*)^2$ ;
$(0, p^2-p, 2p+1)$	(D2) where $-\nu \in (F_p^*)^2$ ; (D3) where $-\nu \in (F_p^*)^2$ ; (D11) where $-\nu \in (F_p^*)^2$ ; (D15)
$(0, p^2-1, p+2)$	(D17), (D18)
$(0, p^2+1, p)$	(D19)
$(0, 1, p)$	(E1)–(E7)
$(0, 1, p^2+p)$	(E8)–(E10)
$(1, 0, p)$	(F1)–(F8); (G1)–(G12)
$(p+1, 0, p^2)$	(H1)–(H3)
$(1, 0, p^2+p)$	(H4)–(H10), (I1)–(I11)
$(p+1, 0, p^3+p^2)$	(J1)–(J5)
$(1, 0, p^3+p^2+p)$	(J6)–(J9)
$(0, 0, p+1)$	(K1)–(K5); (L1)–(L2); (M1)–(M62)
$(0, 0, p^2+p+1)$	(N1)–(N26)
$(0, 0, p^3+p^2+p+1)$	(O1)–(O6)

Table 13: The number of  $\mathcal{A}_0$ -,  $\mathcal{A}_1$ -,  $\mathcal{A}_2$ -subgroups of index  $p$  in  $\mathcal{A}_3$ -groups

$\alpha_1(G)$	types of $\mathcal{A}_3$ -groups $G$
$p^2$	(F1)–(F8)
$p^2 + 1$	(E1), (E3)–(E7)
$p^2 + p - 1$	(A1)–(A6)
$p^2 + p$	(C1)–(C6), (C8)–(C9), (C11), (C15), (C17), (K3)–(K5)
$p^2 + p + 1$	(K1)–(K2)
$p^2 + 2p$	(C16)
$2p^2 - 1$	(B1), (B2) where $n = m = 1$ , (B3) where $n = 1$ , (B4), (B5) where $n = 1$
$2p^2$	(M48)–(M53), (M58)–(M62), (N25)–(N26)
$2p^2 + p - 1$	(B7) where $l = 1$ , (B12) where $l = 1$ , (B13) where $p > 2$ and $l = m = 1$ , (B20) (C7), (C10), (C12)–(C14)
$2p^2 + p$	(D2) where $-\nu \notin (F_p^*)^2$ ; (D3) where $-r \notin (F_p^*)^2$ ; (D5); (D11) where $-\nu \notin (F_p^*)^2$ ; (M54)–(M57)
$3p^2 + 1$	(D19)
$3p^2 + p - 2$	(B8) where $l = 1$
$3p^2 + p$	(L1)–(L2)
$p^3$	(G1)–(G12), (I1)–(I9)
$p^3 + 1$	(E2), (E8)
$p^3 + p^2 - p$	(B2) where $m > 1 = n$ or $n > 1 = m$ ; (B3) where $n \geq 2$ , (B5) where $n \geq 2$
$p^3 + p^2$	(B7) where $l \geq 2$ , (B10), (B12) where $l \geq 2$ , (B13) where $l \geq 2$ and $m = 1$ (B14) where $m = 1$ or $n = 1$ , (B16), (B18), (B18) where $m = 1$ , (B19) where $m = 1$ and $p > 2$ (B19) where $p = 2$ and $m = 1$ and $l > 1$ , (I10)–(I11), (M1)–(M19), (M40)–(M43), (M47) (N6)–(N13), (N15)–(N17), (N19)–(N21)
$p^3 + p^2 + p$	(M38)–(M39)
$p^3 + 2p^2 - p$	(N14), (N18)
$p^3 + 2p^2 - 1$	(B6), (B9), (B11), (B13) where $p = 2$ and $l = m = 1$ (B17) where $l = 1$ , (B19) where $p = 2$ and $l = m = 1$
$p^3 + 2p^2$	(D1), (D4), (D6)–(D10), (D12)–(D14), (D16), (M20)–(M36), (M44)–(M46), (N22)–(N24)
$p^3 + 2p^2 + p$	(M37)
$p^3 + 3p^2 - 1$	(D17)–(D18)
$2p^3 + p^2 - p$	(B8) where $l \geq 2$ , (B13) where $m = 2$ , (B14) where $m = n = 2$ , (B15) (B17) where $l \geq 2$ , (B18) where $m = 2$ , (B19) where $m = 2$
$2p^3 + 2p^2 - p$	(D2) where $-\nu \in (F_p^*)^2$ ; (D3) where $-\nu \in (F_p^*)^2$ ; (D11) where $-\nu \in (F_p^*)^2$ ; (D15)
$p^4$	(H1)–(H3), (J1)–(J5)
$p^4 + p^2$	(O1)–(O4)
$p^4 + p^3 + 1$	(E9)–(E10)
$p^4 + p^3$	(H4)–(H10), (J6)–(J9)
$p^4 + p^3 + p^2$	(N1)–(N5), (O6)
$p^4 + p^3 + p^2 + p$	(O5)

Table 14: The number of  $\mathcal{A}_1$ -subgroups in  $\mathcal{A}_3$ -groups

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